ROBUSTNESS OF FINITE ELEMENT METHODS
FOR A MODEL PARAMETER DEPENDENT PROBLEM

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Summary

A convergence analysis is presented for standard and mixed finite element discretizations of a model system of equations for a transversely loaded beam. The equations depend parametrically on the beam thickness and the emphasis of the analysis is on the robustness of the methods with respect to this parameter. The mixed methods are shown to be far more robust than the standard methods employing elements of the same degree. Moreover they entail no additional computational expense. Computational results are included to illustrate the main results.

1. Introduction

Most numerical software is intended to apply to a class of computational problems. For example, programs in structural mechanics may permit a wide variety of geometries, materials, loadings, etc. For this reason a highly desirable quality of a numerical method is that it be reliably accurate and efficient for the entire class of problems to which it may reasonably be applied. This quality may be referred to as *robustness* of the method.

One quite general way to analyze the robustness of a method is to parametrize the class or a subclass of relevant problems and investigate the uniformity (or lack of uniformity) of convergence of the approximation error with respect to variation of the parameters. Familiar examples of such parametrizations are given by the material coefficients in structural calculations and by the Reynolds number in fluid flow simulations. In this paper we investigate the robustness of two families of finite element methods for a beam model with respect to the parameter of beam thickness. A standard linear finite element method is found to be not robust at all and the analagous method using higher degree elements to have fair robustness properties. A family of mixed methods are defined which are completely robust with respect to beam thickness for elements of all degrees. Moreover these mixed methods can be implemented as a modification of the standard method with an integration rule of reduced accuracy, thus realizing the superior robustness without extra computational cost or programming effort.

The results presented parallel published numerical results for plate and shell computations.[2,3,4,6,7,9,10] These however have not been fully analyzed mathematically. The analysis for the beam which is outlined here has been justified in full detail by the author.[1]

2. Notations

For functions $f(x)$ and $g(x)$ defined for $0 \leq x \leq 1$ we let $<f,g>$ denote $\int_0^1 f(x)g(x)dx$ and set $\|f\| = \sqrt{<f,f>}$. For $r = 0,1,2,\ldots,$ $f^{(r)}$ denotes $\dfrac{d^r f}{dx^r}$, $f' = f^{(1)}$, $\|f\|_r = (\|f\|^2 + \|f'\|^2 + \ldots + \|f^{(r)}\|^2)^{\frac{1}{2}}$, and H^r is the space of functions f for which $\|f\|_r$

is finite. The space $\overset{\circ}{H}^1 = \{f \in H^1 | f(0) = f(1) = 0\}$.

The notation $M^r_{-1}(\Delta)$ refers to the space of all (possibly discontinuous) piecewise polynomials of degree at most r subordinate to a partition Δ of the unit interval. The subspace of M^r_{-1} consisting of continuous functions vanishing at 0 and 1 is denoted $\overset{\circ}{M}^r$.

3. The beam model

The beam model we consider was formulated by Timoshenko[8]. We seek two function $\phi_d(x)$ and $\omega_d(x)$ defined for $0 \leq x \leq 1$ satisfying

$$- \phi_d'' + d^{-2}(\phi_d - \omega_d') = 0 ,$$

$$d^{-2}(\phi_d - \omega_d') = g ,$$

$$\phi_d(0) = \phi_d(1) = \omega_d(0) = \omega_d(1) = 0 .$$

A variational statement is the problem

$$\left. \begin{array}{l} \text{find } \phi_d \, , \, \omega_d \in \overset{\circ}{H}^1 \text{ such that} \\[2mm] <\phi_d',\psi'> + d^{-2}<\phi_d - \omega_d', \psi - \nu'> = <g,\nu> \text{ for all } \psi \, , \, \nu \in \overset{\circ}{H}^1 . \end{array} \right\} (S_d)$$

This model may be derived from the equations of plane linear elasticity by dimensional reduction as follows. Let an undisplaced plane body occupying the region $\{0 \leq x \leq 1 , \frac{-d}{2} \leq y \leq \frac{d}{2}\}$ be subject to a smooth nonzero vertical body force $- d^2 g(x)$. Project in energy the resulting displacement field into the space of displacements which vary linearly with y . Assuming convenient values for the material constants it is easily seen that the projected displacement field is $(-y\phi_d(x), \omega_d(x))$ where ϕ_d and ω_d are defined by (S_d) . Physically ω_d represents the vertical displacement of the midline and ϕ_d the rotation of the cross section.

4. The standard finite element method

Let $\Delta = \{0 = x_0 < x_1 < \ldots < x_n = 1\}$ be a mesh of meshsize $h_\Delta = \max(x_i - x_{i-1})$, and let $r \geq 1$. A straightforward finite element discretization of (S_d) is

find $\phi_{d\Delta}$, $\omega_{d\Delta} \in \mathring{M}^r(\Delta)$ such that

$$\langle\phi_d',\psi'\rangle+d^{-2}\langle\phi_d-\omega_d',\psi-\nu'\rangle=\langle g,\nu\rangle \text{ for all } \psi,\nu\in\mathring{M}^r(\Delta). \left.\right\}(S_{d\Delta})$$

It is not difficult to show that for fixed beam thickness the approximations $\phi_{d\Delta}$ and $\omega_{d\Delta}$ converge to ϕ_d and ω_d at the best possible rate with respect to h_Δ in both H^0 and H^1 .

__Theorem 1.__ There exist constants c_d^i depending on d but independent of g and Δ such that

$$\|\phi_d-\phi_{d\Delta}\| \le c_d^1 h_\Delta^{r+1}\|g\|_{r-1} ,$$

$$\|\phi_d-\phi_{d\Delta}\|_1 \le c_d^2 h_\Delta^r\|g\|_{r-1} ,$$

$$\|\omega_d-\omega_{d\Delta}\| \le c_d^3 h_\Delta^{r+1}\|g\|_{r-1} ,$$

$$\|\omega_d-\omega_{d\Delta}\|_1 \le c_d^4 h_\Delta^r\|g\|_{r-1} . \qquad \square$$

The constants c_d^i cannot be taken to be independent of d . Indeed for linear elements (r=1) $\lim\limits_{d\downarrow 0} c_d^i = +\infty$ for i=1,2,3,4 and if $r \ge 2$ the same is true for i=1,2,3 . Of course the fact that the constant c_d^i does not remain bounded as d decreases to zero does not necessarily mean that the corresponding error in Theorem 1 does not tend to zero uniformly in d , only that it does not do so with as high a rate of convergence as indicated in the theorem for fixed d . In the case of linear elements, however, there is in fact no uniform convergence at all.

__Theorem 2.__ Let r=1 . For any mesh Δ and any positive number $\sigma < 1$

$$\|\phi_d-\phi_{d\Delta}\| > \sigma\|\phi_d\|$$

and

$$\|\omega_d-\omega_{d\Delta}\| > \sigma\|\omega_d\|$$

for all sufficiently small d . $\qquad \square$
This theorem suggests – correctly – that the method $(S_{d\Delta})$ is not practicable for thin beam calculations.

If r > 1 the method *does* converge uniformly in d . However three of the four rates of convergence given in Theorem 1 are *reduced*.

__Theorem 3.__ Let $r \ge 2$. There exist constants K^i independent of d , g , and Δ such that

$$\|\phi_d-\phi_{d\Delta}\| \le K^1 h_\Delta^r\|g\|_{r-1} ,$$

$$\|\phi_d-\phi_{d\Delta}\|_1 \le K^2 h_\Delta^{r-1}\|g\|_{r-1} ,$$

$$\|\omega_d-\omega_{d\Delta}\| \le K^3 h_\Delta^r\|g\|_{r-1} ,$$

$$\|\omega_d-\omega_{d\Delta}\|_1 \le K^4 h_\Delta^r \|g\|_{r-1} . \qquad \square$$

5. A mixed finite element method

Let $\zeta_d = d^{-2}(\phi_d-\omega_d')$. We present a variational problem equivalent to (S_d) but involving ζ_d as an additional independent variable. Since ζ_d represents the shear stress we refer to this formulation as mixed.

Find $\phi_d,\omega_d\in\mathring{H}^1$ and $\zeta_d\in H^0$ such that

$$\langle\phi_d',\psi'\rangle+\langle\zeta_d,\psi-\nu'\rangle=\langle g,\nu\rangle \text{ for all } \psi , \nu\in\mathring{H}^1 \text{ and}$$

$$\langle\phi_d-\omega_d',\eta\rangle-d^2\langle\zeta_d,\eta\rangle=0 \text{ for all } \eta\in H^0 . \left.\right\}(M_d)$$

To discretize this system we again approximate ϕ_d and ω_d in $\mathring{M}^r$ and restrict ψ and ν to $\mathring{M}^r$, and also approximate ζ_d in M_{-1}^{r-1} and restrict to that space. The resulting mixed finite element method is

find $\bar\phi_{d\Delta},\bar\zeta_{d\Delta}\in\mathring{M}^r$ and $\bar\zeta_{d\Delta}\in M_{-1}^{r-1}$ so that

$$\langle\bar\phi_{d\Delta}',\psi'\rangle+\langle\bar\zeta_{d\Delta},\psi-\nu'\rangle=\langle g,\nu\rangle \text{ for all } \psi , \nu\in\mathring{M}^r \text{ and}$$

$$\langle\bar\phi_{d\Delta}-\bar\omega_{d\Delta},\eta\rangle-d^2\langle\zeta_{d\Delta},\eta\rangle=0 \text{ for all } \eta\in M_{-1}^{r-1}. \left.\right\}(M_{d\Delta})$$

The author has proven an abstract stability theorem which applies to variational systems of the form of (M_d) and $(M_{d\Delta})$.[1] It follows from this theorem that there is a unique solution to each system and that

$$\|\phi_d-\bar\phi_{d\Delta}\|_1 + \|\omega_d-\bar\omega_{d\Delta}\|_1 + \|\zeta_d-\bar\zeta_{d\Delta}\|$$

$$\le C[\inf_{\phi\in\mathring{M}^r}\|\phi_d-\phi\|_1 + \inf_{\omega\in\mathring{M}^r}\|\omega_d-\omega\|_1 + \inf_{\zeta\in M_{-1}^{r-1}}\|\zeta_d-\zeta\|]$$

where the constant C is independent of d , g , and Δ . Combining this result with a duality argument and regularity and approximability results for the solution of (M_d) we get the following theorem which shows that the best possible rates of convergence which can occur for fixed d hold for this method *uniformly in d* .

__Theorem 4.__ There exists a constant C independent of d , g , and Δ such that

$$\|\phi_d-\bar\phi_{d\Delta}\| \le Ch_\Delta^{r+1}\|g\|_{r-1} ,$$

$$\|\phi_d-\bar\phi_{d\Delta}\|_1 \le Ch_\Delta^r\|g\|_{r-1} ,$$

$$\|\omega_d-\omega_{d\Delta}\| \le Ch_\Delta^{r+1}\|g\|_{r-1} ,$$

$$\|\omega_d-\bar\omega_{d\Delta}\|_1 \le Ch_\Delta^r\|g\|_{r-1} . \qquad \square$$

Thus the mixed methods are entirely robust with respect to beam thickness and – comparing with Theorems 2 and 3 – are much superior to the standard methods in this respect.

For practical purposes it is important to note that the methods $(M_{d\Delta})$ are examples of mixed methods for which the additional variable can be eliminated analytically at the discretized level, resulting in a reduced integration finite element scheme.[5] Specifically $\phi_{d\Delta}$ and $\omega_{d\Delta}$ may be defined exactly as $\phi_{d\Delta}$ and $\omega_{d\Delta}$ in $(S_{d\Delta})$ except that the integrals occurring in the stiffness matrix must be replaced by r point composite Gauss quadratures. In this way the additional robustness of the mixed methods may be achieved without additional computational expense.

6. Illustrative computational results

Graphs 1-12 present the results of computations which illustrate the above analysis. In each graph the errors in ϕ_d, ω_d, ϕ_d', and ω_d' are plotted versus the meshsize on logarithmically transformed axes. The results for the methods $(S_{d\Delta})$ and $(M_{d\Delta})$ are shown for linear elements (graphs 1-3 and 4-6, respectively) and quadratic elements (graphs 7-9 and 10-12, respectively). In all cases the load is constant and the mesh is uniform. For each method graphs are given for three cases: d held constant and relatively large (d=.1), d decreasing with the meshsize (d=h$^{3/2}$ for linear elements d=h^2 for quadratic elements), and d constant and relatively small (d=.001). In the first case - which does not tax the robustness of the method - both methods perform well, illustrating the optimal rates of convergence stated in Theorems 1 and 4. The second case illustrates the nonuniform nature of the convergence of the standard method. The lack of uniform convergence of the linear method (Theorem 2) and the reduced rates of convergence of the quadratic method (Theorem 3) are clearly visible in graphs 2 and 8 respectively. The importance of this behavior for practical computations is highlighted in graphs 3 and 9. For small thickness it is clearly the uniform rates which govern the convergence until h_Δ is extremely small. In contrast the robustness of the mixed methods is graphically illustrated by the fact that the error is essentially unaffected by d and graphs 4-6 are virtually identical, as are graphs 10-12.

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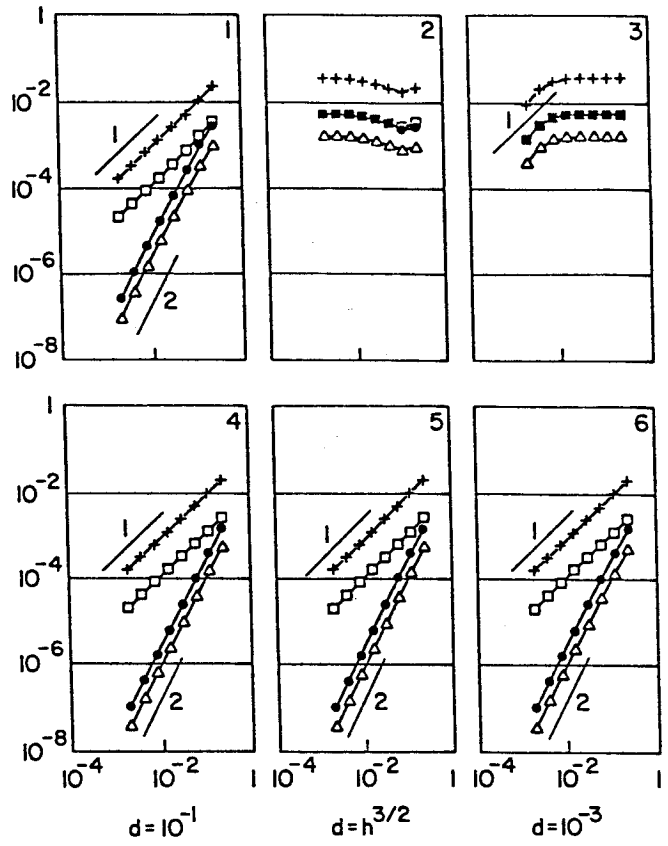
Error in linear finite elements as function of h .

Graphs 1-3:   standard method   ● H^0 norm of error in ϕ_d

+ H^0 norm of error in ϕ_d'

Graphs 4-6:   mixed method   Δ H^0 norm of error in ω_d

□ H^0 norm of error in ω_d'

Numbered line segments are drawn at marked slopes.
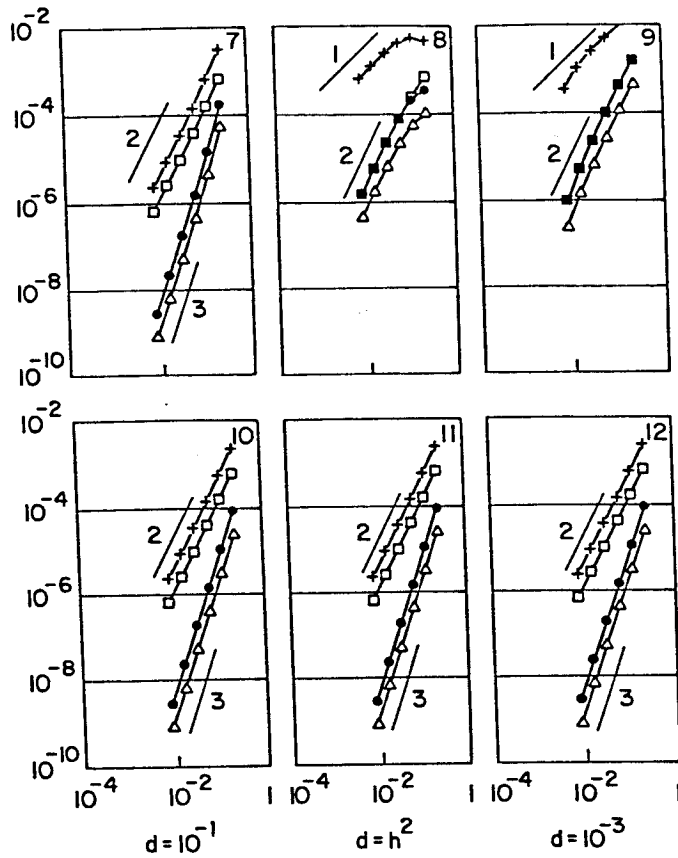
Error in quadratic finite elements as function of h   .

Graphs 7-9:   standard method   ● H^0 norm of error in ϕ_d

                                    + H^0 norm of error in ϕ'_d

Graphs 10-12: mixed method   Δ H^0 norm of error in ω_d

                                    □ H^0 norm of error in ω'_d

Numbered line segments are drawn at marked slopes.