Answer to the first homework problem

I am providing you with the answer to the first homework problem to give you an idea of the level of detail and style of presentation I would like to see from you.

1. Consider a 9-point difference approximation Δ_h^* to the Laplacian with stencil as shown. Show how to choose the constants α, β, and γ so that the scheme $\Delta_h^* u_h = f$ is consistent to fourth order with the equation $\Delta u = f$.

$$\frac{1}{h^2}\begin{pmatrix} & & \alpha & & \\ & & \beta & & \\ \alpha & \beta & \gamma & \beta & \alpha \\ & & \beta & & \\ & & \alpha & & \end{pmatrix}$$

Answer: Let (x, y) be a grid point and u a smooth function. The difference scheme is

$$\Delta_h^* u(x,y) = \frac{1}{h^2}\{\alpha[u(x-2h,y)+u(x+2h,y)+u(x,y-2h)+u(x,y+2h)] \quad (1)$$
$$+\beta[u(x-h,y)+u(x+h,y)+u(x,y-h)+u(x,y+h)]+\gamma u(x,y)\}.$$

We wish to determine values of α, β, and γ so that

$$\Delta_h^* u(x,y) = \Delta u(x,y) + O(h^4). \quad (2)$$

By Taylor's theorem we have

$$u(x+h,y) = u + h\frac{\partial u}{\partial x} + \frac{h^2}{2}\frac{\partial^2 u}{\partial x^2} + \frac{h^3}{6}\frac{\partial u^3}{\partial x^3} + \frac{h^4}{24}\frac{\partial u^4}{\partial x^4} + \frac{h^5}{120}\frac{\partial u^5}{\partial x^5} + O(h^6),$$

where the functions u, $\partial u/\partial x$, etc., on the right hand side are evaluated at (x, y). Replacing h by $-h$, $2h$, and $-2h$ in this equation, we obtain expansions for $u(x-h, y)$, $u(x+2h, y)$ and $u(x-2h, y)$, respectively:

$$u(x-h,y) = u - h\frac{\partial u}{\partial x} + \frac{h^2}{2}\frac{\partial^2 u}{\partial x^2} - \frac{h^3}{6}\frac{\partial u^3}{\partial x^3} + \frac{h^4}{24}\frac{\partial u^4}{\partial x^4} - \frac{h^5}{120}\frac{\partial u^5}{\partial x^5} + O(h^6),$$
$$u(x+2h,y) = u + 2h\frac{\partial u}{\partial x} + \frac{4h^2}{2}\frac{\partial^2 u}{\partial x^2} + \frac{8h^3}{6}\frac{\partial u^3}{\partial x^3} + \frac{16h^4}{24}\frac{\partial u^4}{\partial x^4} + \frac{32h^5}{120}\frac{\partial u^5}{\partial x^5} + O(h^6),$$
$$u(x-2h,y) = u - 2h\frac{\partial u}{\partial x} + \frac{4h^2}{2}\frac{\partial^2 u}{\partial x^2} - \frac{8h^3}{6}\frac{\partial u^3}{\partial x^3} + \frac{16h^4}{24}\frac{\partial u^4}{\partial x^4} - \frac{32h^5}{120}\frac{\partial u^5}{\partial x^5} + O(h^6).$$

Combining these expressions we find that

$$\frac{1}{h^2}\{\alpha[u(x-2h,y)+u(x+2h,y)]+\beta[u(x-h,y)+u(x+h,y)]+\frac{\gamma}{2}u(x,y)\}$$
$$= h^{-2}(2\alpha+2\beta+\frac{\gamma}{2})u + (8\alpha+2\beta)\frac{1}{2}\frac{\partial^2 u}{\partial x^2} + (32\alpha+2\beta)\frac{h^2}{24}\frac{\partial^4 u}{\partial x^4} + O(h^4).$$

(Note that the odd-order derivative terms cancel.)

In exactly the same way,

$$\frac{1}{h^2}\{\alpha[u(x,y-2h)+u(x,y+2h)]+\beta[u(x,y-h)+u(x,y+h)]+\frac{\gamma}{2}u(x,y)\}$$
$$= h^{-2}(2\alpha+2\beta+\frac{\gamma}{2})u + (8\alpha+2\beta)\frac{1}{2}\frac{\partial^2 u}{\partial y^2} + (32\alpha+2\beta)\frac{h^2}{24}\frac{\partial^4 u}{\partial y^4} + O(h^4).$$

Adding these two expressions then gives

$$\Delta_h^* u = h^{-2}(4\alpha+4\beta+\gamma)u + (8\alpha+2\beta)\frac{1}{2}\Delta u + (32\alpha+2\beta)\frac{h^2}{24}\left(\frac{\partial^4 u}{\partial x^4}+\frac{\partial^4 u}{\partial y^4}\right) + O(h^4).$$

Therefore (2) holds if and only if

$$4\alpha + 4\beta + \gamma = 0, \quad 8\alpha + 2\beta = 2, \quad 32\alpha + 2\beta = 0.$$

This linear system has the unique solution $\boxed{\alpha = -1/12, \quad \beta = 4/3, \quad \gamma = -5}$.

Thus the 9-point stencil which is consistent to fourth order with the Laplacian is

$$\frac{1}{12h^2}\begin{pmatrix} & & -1 & & \\ & & 16 & & \\ -1 & 16 & -60 & 16 & -1 \\ & & 16 & & \\ & & -1 & & \end{pmatrix}.$$