Do a computer study of **one** of the following problems using FEniCS. Use FEniCS to find some interesting phenomena about the solution of the PDE, the performance of the finite element method, or, preferably, both. Use the problem description as a guideline, but feel free to investigate additional directions. Hand in a well-organized report including graphics, a listing of your program(s), numerical results, and discussion.

Note: if you would like to substitute a problem of your own devising rather than follow one of the problems below, please discuss it with me.

1. Consider the Laplace equation $\Delta u = 0$ on the unit square with homogeneous Dirichlet boundary conditions everywhere on the boundary except on the bottom half of the left edge (i.e., on the interval $x = 0$, $0 < y < 1/2$), where the Neumann boundary condition $\partial u/\partial n = 1$ is imposed.

Investigate the solution. Plot u, $\partial u/\partial x$, and $\partial u/\partial y$. There are singularities at the end points of the interval where Neumann conditions are applied. Investigate how u, $\partial u/\partial x$, and $\partial u/\partial y$ behave near these points.

Determine the value of $\partial u/\partial x$ at the point $(.25, .25)$ to at least 4 decimal places, using a fine mesh and finite elements of higher degree. Then recompute this quantity using Lagrange elements of degree 1 and meshes of size $n \times n$ with $n = 4, 8, 16, 32, 64$. What is the apparent rate of convergence?

2. Consider the problem

$$-\epsilon \Delta u + \frac{\partial u}{\partial x} = 1 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

where Ω is the unit square (or another domain, if you prefer). For $\epsilon \approx 1$, this is straightforward to solve, but for ϵ very small, it is more difficult. Study both $\epsilon = 1$ and $\epsilon = 0.01$ using piecewise linear finite elements with meshes of size $n \times n$ with $n = 4, 8, 16, \ldots$. What about quadratic (or higher) degree finite elements?

3. In a manner similar to the above problems, study the nature of the solution and the performance of the finite element method for some problems with localized near-degeneracy in the coefficient:

$$-\operatorname{div}(a \operatorname{grad} u) = 1 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

taking the coefficient function a to be

$$a(x, y) = 1 - .99 \exp(-100[(x - .25)^2 + (y - .3)^2])$$

(which is nearly 0 in a small part of the domain) and then to be

$$a(x, y) = 1 + 99 \exp(-100[(x - .25)^2 + (y - .3)^2])$$

(which is very large in a small part of the domain). Compute the solutions and interpret what you see, and discuss the performance of the finite element method with linear elements.