1. Consider solving the system $Au = f$ where A is SPD, so the solution minimizes $u^T Au/2 - f^T u$. Let u_0 and s_0 be *any* vectors (initial iterate and search direction), and define u_1 by exact line search. Prove that the error $u_* - u_1$ is A-orthogonal to s_0.

2. Suppose that A is a 100×100 SPD matrix but has eigenvalues of high multiplicity, so that there are only 3 distinct eigenvalues altogether. Show that the conjugate gradient method converges in just 3 iterations.

3. Consider the Richardson iteration with parameter ω applied to an SPD matrix A. Recall that it converges for $0 < \omega < 2/\lambda_{\max}$ (where $\lambda_{\max}$ is the largest eigenvalue of A). Consider the cases of (a) ω greater than but close to 0 (e.g., $\omega \approx 10^{-6}$); (b) ω less than but close to $2/\lambda_{\max}$ (e.g., $\omega \approx 1.99/\lambda_{\max}$); and (c) ω near $1/\lambda_{\max}$. Investigate in which, if any, of these cases, the Richardson iteration has the smoothing property and so would be a useful smoother for multigrid.