Notation: A compact notation for a finite difference scheme is to display its coefficients in a *stencil*. For example, the usual 5-point Laplacian Δ_h has the stencil shown on the left of this display:

$$\frac{1}{h^2}\begin{pmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{pmatrix} \qquad \frac{1}{h^2}\begin{pmatrix} & & \alpha & & \\ & & \beta & & \\ \alpha & \beta & \gamma & \beta & \alpha \\ & & \beta & & \\ & & \alpha & & \end{pmatrix} \qquad \frac{1}{h^2}\begin{pmatrix} \alpha & \beta & \alpha \\ \beta & \gamma & \beta \\ \alpha & \beta & \alpha \end{pmatrix}$$

1. Consider a 9-point difference approximation Δ_h^* to the Laplacian with stencil as given in the center of the display above. Show how to choose the constants α, β, and γ so that the scheme $\Delta_h^* v = f$ is consistent to fourth order with the equation $\Delta u = f$.

2. Next consider a 9-point approximation of the Laplacian with stencil as shown on the right of the display. Show that there is no choice of constants α, β, and γ so that this scheme is fourth order accurate. However show that the coefficients can be chosen to give a fourth order scheme of the form $\Delta_h^\dagger v = R_h f$ where R_h is a difference operator with stencil
$$\frac{1}{12}\begin{pmatrix} & 1 & \\ 1 & 8 & 1 \\ & 1 & \end{pmatrix}.$$

3. Show that with the same choice of coefficients as in the last problem the scheme $\Delta_h^\dagger v = 0$ is a *sixth* order accurate approximation of the homogeneous equation $\Delta u = 0$.

4. Consider the solution of the Poisson equation with zero Dirichlet boundary conditions on a hexagon. Develop a 7-point Laplacian using mesh points lying at the vertices of a grid of equilateral triangles. Prove L^∞ convergence of the method and exhibit the rate of convergence.