1. Prove that Euler's method is stable with respect to perturbations in the initial data y_0 and the function f. That is, prove that if y_n is defined by Euler's method and $\bar{y}_n$ is defined by the perturbed equations:

$$\bar{y}_{n+1} = \bar{y}_n + h\bar{f}(t_n, \bar{y}_n), \quad \bar{y}_0 \text{ given},$$

   then $|y_n - \bar{y}_n| \le C_1|y_0 - \bar{y}_0| + C_2\|f - \bar{f}\|_{L^\infty(I\times\mathbb{R})}$. (State precisely the hypotheses needed and give explicit formulas for C_1 and C_2.)

2. State precisely and prove an asymptotic error estimate for the trapezoidal method.

3. Consider a consistent linear multistep methods $y_{n+1} + \sum_{j=0}^{k} a_j y_{n-j} = h\sum_{j=-1}^{k} b_j f_{n-j}$ for which the coefficients $a_j \le 0$, $j = 0, \dots, k$ (there are many such methods). a) Prove that all such consistent methods satisfy the root condition, and so are stable. b) Give an elementary proof that all such methods are convergent (without invoking the Dahlquist theory).

4. Show that a Runge–Kutta method of q stages always has order $\le q$. (Hint: consider an appropriate simple ODE for which you know the exact solution.)

5. Derive the general 3-stage third order explicit Runge–Kutta method.

6. A COMPUTATIONAL PROBLEM WILL BE ADDED VERY SOON. CHECK BACK!