The Single Layer Heat Potential and Galerkin Boundary Element Methods for the Heat Equation

by

Patrick James Noon

Dissertation submitted to the Faculty of the Graduate School
of The University of Maryland in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
1988

Advisory Committee:

Professor D. N. Arnold
Professor I. Babuska
Professor R. B. Kellog
Professor F. W. J. Olver
Professor J. R. Dorfman

Contents

Abstract

Title of Dissertation: The Single Layer Heat Potential and Galerkin Boundary Element Methods for the Heat Equation

Patrick James Noon, Doctor of Philosophy, 1988.

Dissertation directed by: Douglas N. Arnold, Associate Professor, Applied Mathematics Department.

We study Galerkin boundary element discretizations of the single layer heat potential operator equation

$$\mathcal{K}_1 q := \int_0^t \int_\Gamma K(x-y, t-t')q(y,t')dydt' = F(x,t), \qquad x \in \Gamma,\ t > 0, \quad (0.1)$$

where K denotes the fundamental solution

$$K(x,t) = \begin{cases} \dfrac{\exp(-|x|^2/4t)}{(4\pi t)^{3/2}} & x \in \mathbb{R}^3,\ t > 0 \\ 0 & x \in \mathbb{R}^3,\ t < 0 \end{cases}.$$

We first formulate a well-posedness theory for (0.1) and show that for each F in the anisotropic Sobolev space $H^{1/2,1/4}(\Gamma, \mathbb{R}_+)$, there exists a unique solution q in its dual space $H^{-1/2,-1/4}(\Gamma, \mathbb{R}_+)$ which depends continuously on the data in the sense that

$$\|q\|_{H^{-1/2,-1/4}(\Gamma,\mathbb{R}_+)} \leq c\|F\|_{H^{1/2,1/4}(\Gamma,\mathbb{R}_+)}.$$

Moreover, we show that $\mathcal{K}_1$ satisfies a coercivity estimate

$$\text{Re}\ \langle q, \mathcal{K}_1 q\rangle \geq c\|q\|^2_{H^{-1/2,-1/4}(\Gamma,\mathbb{R}_+)}.$$

We next develop a regularity theory for the mapping $\mathcal{K}_1$ and show that $\mathcal{K}_1$ (in the scale of anisotropic Sobolev spaces $H^{r,s}(\Gamma, \mathbb{R}_+)$ for $r, s \geq 0$) may be regarded as an operator which increases regularity by one spatial derivative and one-half time derivative. These results provide a basis for our subsequent analysis of a class of Galerkin discretizations methods based on test and trial spaces of piecewise polynomials. We show optimal convergence in the energy norm $H^{-1/2,-1/4}(\Gamma, \mathbb{R}_+)$ and investigate the rate of convergence in $L^2(\Gamma \times \mathbb{R}_+)$. Finally, to test our conclusions, we present numerical examples.

1 Introduction

The classical method of boundary integral equations uses specifically defined solutions called layered potentials and reduces the given boundary value problem into an integral equation of the second kind. Thus, a double layer potential is used to treat Dirichlet problems, whereas a single layer potential is used to solve Neumann problems. The overwhelming reason for these selections is the well known results on the unique solvability of second kind integral equations. Besides being well posed in a variety of spaces, second kind integral equations are often well suited, (in regards to both implementation and analysis), to various approximate methods such as Galerkin or collocation type methods.

Although the layered potential approach is more commonly associated with elliptic problems, it also has a long history in the study of parabolic boundary value problems. Holmgren [21] initially introduced the heat potentials in two variables (i.e., one time and one space variable) and used them to show the solvability of the heat equation. Gevrey [18] subsequently extended the argument to more general parabolic problems of two variables. Generalizations to problems in several space dimensions were slow to appear since the kernel of the second kind Volterra integral equation which arises was not fully understood. Pogorzelski [35, 36, 37, 38] showed the basic solvability of this integral equation in arbitrarily many space dimension on smooth manifolds. His arguments helped to establish the basic well-posedness of a wide variety of parabolic problems.

Currently, the application of boundary element methods to parabolic problems is being actively considered. Essentially, boundary element methods refer to numerical solutions of the integral equations encountered in the layered potential method. In contrast to the classical approach, however, the integral equation typically used for numerical purposes is the so called direct integral equation of heat conduction. In the case of problems with Dirichlet boundary conditions, this equation results in a first kind Volterra integral equation. Though this approach is often used in practice [8], [12], [26], [42], [43], the basic convergence theory behind it has yet to be given. In this paper, we analyze the direct integral equation method applied to an initial-Dirichlet boundary value problem for the heat equation.

We start by recalling the direct integral equation. Let $u(x,t)$ solve the initial-Dirichlet boundary value problem

$$\begin{aligned} \frac{\partial u}{\partial t}(x,t) - \Delta u(x,t) &= 0, \qquad x \in \Omega,\ t > 0, &(1.1)\\ u(x,0) &= f(x), \qquad x \in \Omega, &(1.2)\\ u(x,t) &= g(x,t), \qquad x \in \Gamma,\ t > 0, &(1.3)\end{aligned}$$

where Ω denotes a bounded, open set in $\mathbb{R}^3$. Set $K(x,t)$ equal to the fundamental solution to the heat equation, i.e.,

$$K(x,t) = \begin{cases} \dfrac{\exp(-|x|^2/4t)}{(4\pi t)^{3/2}} & x \in \mathbb{R}^3,\ t > 0 \\ 0 & x \in \mathbb{R}^3,\ t < 0 \end{cases}.$$

Then, a simple application of Green's theorem (see [34, pp. 42–43]) shows that the solution u to (1.1–1.3) must satisfy the integral equation

$$\begin{aligned} u(x,t) \;=\; & \int_0^t \int_\Gamma \left[K(x-y,t-t')\frac{\partial u}{\partial \mathbf{n}_y}(y,t') - \frac{\partial K}{\partial \mathbf{n}_y}(x-y,t-t')g(y,t') \right] dy dt' \\ & + \int_\Omega K(x-x',t)f(x')dx', \qquad x \in \Omega, t > 0, \end{aligned} \tag{1.4}$$

where Γ denotes the boundary of Ω and $\mathbf{n}_y$ the unit outward normal derivative to Γ at y. We will assume that Γ is a smooth C^2 surface. Observe how the direct integral equation (1.4) relates u throughout $\Omega \times \mathbb{R}_+$ to its initial and boundary data.

The first term of (1.4), (where we have set $q = \partial u/\partial \mathbf{n}_y$)

$$U_1(x,t) := \int_0^t \int_\Gamma K(x-y,t-t')q(y,t')dydt', \qquad x \in \mathbb{R}^3\backslash\Gamma,\ t > 0, \tag{1.5}$$

is called the single layer heat potential with density q. Similarly, the second term of (1.4)

$$U_2(x,t) := \int_0^t \int_\Gamma \frac{\partial K}{\partial \mathbf{n}_y}(x-y,t-t')g(y,t)dydt', \qquad x \in \mathbb{R}^3\backslash\Gamma,\ t > 0 \tag{1.6}$$

is the double layer heat potential with density g. Assuming that q and g are continuous on $\Gamma \times \mathbb{R}_+$, each of these potentials defines a C^∞ function on $x \in \mathbb{R}^3\backslash\Gamma$ and $t > 0$ which satisfy the heat equation there and vanish for

$t = 0$. They also satisfy jump conditions (cf., [17, p. 137]) similar to those satisfied by the corresponding single and double layer electrostatic potentials. As $x_0 \in \mathbb{R}^3 \backslash \Gamma$ tends non-tangentially to $x \in \Gamma$, we have

$$\lim_{x_0 \to x} U_1(x_0, t) = U_1(x, t), \tag{1.7}$$

$$\lim_{x_0 \to x} \frac{\partial U_1}{\partial \mathbf{n}_x}(x_0, t) = \pm \frac{1}{2} q(x, t) + \int_0^t \int_\Gamma \frac{\partial K}{\partial \mathbf{n}_x}(x - y, t - t') q(y, t') dy dt', \tag{1.8}$$

$$\lim_{x_0 \to x} U_2(x_0, t) = \mp \frac{1}{2} g(x, t) + \int_0^t \int_\Gamma \frac{\partial K}{\partial \mathbf{n}_y}(x - y, t - t') g(y, t') dy dt'. \tag{1.9}$$

In these equations, the upper sign holds when the limit is approached from the interior while the lower sign holds when the limit is approached from the exterior.

Letting x in (1.4) tend to the boundary Γ, the jump conditions yield the boundary integral equation

$$\begin{aligned} \frac{1}{2} g(x, t) &= \int_0^t \int_\Gamma [K(x - x', t - t') q(x', t') - \frac{\partial K}{\partial \mathbf{n}_y}(x - y, t - t') g(y, t')] dy dt' \\ &+ \int_\Omega K(x - x', t) f(x') dx', \qquad x \in \Gamma,\ t > 0. \end{aligned} \tag{1.10}$$

This is a first kind Volterra integral equation for the unknown Neumann data $q = \partial u / \partial \mathbf{n}$ of the form

$$\mathcal{K}_1 q := \int_0^t \int_\Gamma K(x - y, t - t') q(y, t') dy dt' = F(x, t), \qquad x \in \Gamma,\ t > 0, \tag{1.11}$$

where

$$\begin{aligned} F(x, t) &= \frac{1}{2} g(x, t) + \int_0^t \int_\Gamma \frac{\partial K}{\partial \mathbf{n}_y}(x - y, t - t') g(y, t') dy dt' \\ &- \int_\Omega K(x - x', t) f(x') dx', \qquad (x, t) \in \Gamma \times \mathbb{R}_+. \end{aligned}$$

Conversely, if we were studying the Neumann (or Robin) boundary value problem, (1.4) would result in a second kind Volterra integral equation for the unknown Dirichlet data u. For example, the form of this equation for the Neumann problem would be

$$\frac{1}{2} u(x, t) + \mathcal{K}_2 u(x, t) = F(x, t), \qquad x \in \Gamma,\ t > 0 \tag{1.12}$$

where $F(x,t)$ is known and where $\mathcal{K}_2$ denotes the integral operator

$$\mathcal{K}_2 p(x,t) = \int_0^t \int_\Gamma \frac{\partial K}{\partial \mathbf{n}_y}(x-y,t-t')p(y,t')dydt', \qquad x \in \Gamma,\, t>0. \quad (1.13)$$

The theory of (1.12) is well developed. Assuming that Γ is a C^2 surface, Pogorzelski [36] showed that the kernel of (1.13) satisfies the estimate

$$|\frac{\partial K}{\partial \mathbf{n}_y}(x,t)| \leq C_\mu t^{-\mu}|x|^{-n+2\mu} \quad \text{for all} \quad \mu \in (1/2,1), \quad (1.14)$$

and is therefore weakly singular. From (1.14), it follows that $\mathcal{K}_2$ has norm less than 1 on $C(\Gamma \times (0,T))$ for T sufficiently small. Thus, $I+\mathcal{K}_2$ is invertible for sufficiently small T. Since $\mathcal{K}_2$ is of convolution type, however, the existence of $(I+\mathcal{K}_2)^{-1}$ on $C(\Gamma \times [0,T])$ for any finite value of T is easy to show by successively considering $I+\mathcal{K}_2$ over subintervals of small length. The same reasoning applies to $I+\mathcal{K}_2$ on $L^2(\Gamma \times (0,T))$. In [37], Pogorzelski showed that $\mathcal{K}_2$ defines a compact mapping on the space of continuous functions. The compactness of the operator $\mathcal{K}_2$ is relevant to the study of numerical discretizations. In particular, the convergence of a broad class of methods known as projection methods (which include Galerkin and collocation methods) is assured when applied to operators of the form $I+\mathcal{K}$ with $\mathcal{K}$ compact. More recent treatments of (1.12) have focused on the case of unsmooth boundaries. Though $\mathcal{K}_2$ no longer remains compact, the basic solvability of this equation holds in a wide variety of function spaces. For more details, see [14], [27]. A treatment of the numerical solution of this equation has been given in some generality by Costabel, Wendland and Onishi [10].

The theory of equation (1.11), however, is less straightforward then that of (1.12). Indeed, until the recent work of Brown [7], even the basic well posedness of this equation had not been addressed. Our first goal is to establish the well-posedness of (1.11) in such a way to provide a basis for the subsequent analysis of discretizations. We do this by giving a variational interpretation of (1.11), extending to the parabolic setting the argument of Nedelec and Planchard [28] who treated the electrostatic single layer potential in this fashion. The main advantage of this viewpoint is that Galerkin discretizations methods may then be analyzed by standard techniques.

To explain our results, we briefly summarize the argument in [28]. Let $W^1(\mathbb{R}^3)$ denote the closure of $\mathcal{D}(\mathbb{R}^3)$ in the norm

$$\|\phi\|_{W^1(\mathbb{R}^3)} = \|\nabla \phi\|_{L^2(\mathbb{R}^3)}, \qquad \phi \in \mathcal{D}(\mathbb{R}^3).$$

The inner product

$$(u,v)_{W^1(\mathbb{R}^3)} = \int_{\mathbb{R}^3} \nabla u(x) \nabla v(x) dx, \tag{1.15}$$

induces a Hilbert space structure on $W^1(\mathbb{R}^3)$. Even though $W^1(\mathbb{R}^3)$ is not included in $L^2(\mathbb{R}^3)$, we have the dense inclusions

$$\mathcal{D}(\mathbb{R}^3) \subset H^1(\mathbb{R}^3) \subset W^1(\mathbb{R}^3).$$

Consequently, the dual $(W^1(\mathbb{R}^3))^*$ of $W(\mathbb{R}^3)$ is identifiable with a subset of distributions strictly contained in $H^{-1}(\mathbb{R}^3)$. As is customary, we denote this space by $W^{-1}(\mathbb{R}^3)$.

The space $W^1(\mathbb{R}^3)$ is introduced because the natural isomorphism from $W^1(\mathbb{R}^3)$ onto $W^{-1}(\mathbb{R}^3)$ defined by the inner product (1.15) clearly extends the distributional definition of the (negative) Laplacian operator $-\Delta$. This explains why the space $W^1(\mathbb{R}^3)$ has been well studied. Based on Sobolev's inequality [13],

$$\|\theta\|_{L^6(\mathbb{R}^3)} \leq C \|\nabla \theta\|_{L^2(\mathbb{R}^3)}, \quad \text{for all} \quad \theta \in \mathcal{D}(\mathbb{R}^3),$$

it follows that $W^1(\mathbb{R}^3)$ is identical (algebraically and topologically) to the subset of $L^6(\mathbb{R}^3)$ functions whose gradients belong to $L^2(\mathbb{R}^3)$. Moreover, the weighted L^2 estimate,

$$\int_{\mathbb{R}^3} |x|^{-2} |u(x)|^2 dx \leq C \|\nabla u\|^2_{L^2(\mathbb{R}^3)}, \qquad u \in W^1(\mathbb{R}^3), \tag{1.16}$$

shown by Hardy's inequality, shows that $W^1(\mathbb{R}^3)$ is a space of locally integrable functions which differs from the space $H^1(\mathbb{R}^3)$ solely in its permitted behavior at infinity. Consequently, the trace operator γ of restriction from $\mathbb{R}^3$ to Γ extends to a surjection of $W^1(\mathbb{R}^3)$ onto $H^{1/2}(\Gamma)$. These facts enable one to show that the operator $\mathcal{K} = \gamma \circ (-\Delta)^{-1} \circ \gamma^*$ defines an isomorphism of $H^{-1/2}(\Gamma)$ onto $H^{1/2}(\Gamma)$. It remained for Nedelec and Planchard to show that $\mathcal{K}$ extends the classical single layer potential

$$\mathcal{K}q(x) = \frac{1}{4\pi} \int_\Gamma \frac{q(y)\, dy}{|x-y|}, \quad x \in \mathbb{R}^3, \qquad q \in \mathcal{D}(\mathbb{R}^3).$$

For our treatment, it will be necessary to work in the setting of the anisotropic Sobolev spaces (cf., [25, Chapter 4]). For all non-negative real numbers r and s, we use $H^{r,s}$ to denote the Hilbert spaces

$$H^{r,s} = L^2(\mathbb{R}, H^r(\mathbb{R}^3)) \cap H^s(\mathbb{R}, L^2(\mathbb{R}^3)),$$

with associated norm

$$\|u\|^2_{H^{r,s}} = \|u\|^2_{L^2(\mathbb{R},H^r(\mathbb{R}^3))} + \|u\|^2_{H^s(\mathbb{R},L^2(\mathbb{R}^3))}.$$

Analogously, let $H^{r,s}(\Gamma, \mathbb{R}_+)$ denote the Hilbert space

$$H^{r,s}(\Gamma, \mathbb{R}_+) = L^2(\mathbb{R}_+, H^r(\Gamma)) \cap H^s(\mathbb{R}_+, L^2(\Gamma)), \tag{1.17}$$

with

$$\|u\|^2_{H^{r,s}(\Gamma,\mathbb{R}_+)} = \|u\|^2_{L^2(\mathbb{R}_+,H^r(\Gamma))} + \|u\|^2_{H^s(\mathbb{R}_+,L^2(\Gamma))}.$$

The utility of these anisotropic Sobolev spaces in treating the heat equation is evident in the discussion in [25, Chapter 4] and the recent work of [6]. Accordingly, there is a well developed theory of the trace operator on these spaces. Letting γ_+ denote an extension of the restriction operator from functions on $\mathbb{R}^3 \times \mathbb{R}$ to functions on $\Gamma \times \mathbb{R}_+$, the results in [25, Theorem 2.1, p. 11] imply that γ_+ extends to a bounded, linear operator of $H^{r,s}$ onto $H^{r-1/2,(r-1/2)s/r}(\Gamma, \mathbb{R}_+)$ for all $r > 1/2$ and any $s \geq 0$. We give a complete review of this trace theorem in section 2 since we also require an important additional trace result which shows that γ_+ even maps the smaller Sobolev space

$$V = \{u(x,t) \in L^2(\mathbb{R}, H^1(\mathbb{R}^3)) \colon \frac{\partial u}{\partial t}(x,t) \in L^2(\mathbb{R}, H^{-1}(\mathbb{R}^3))\},$$

onto $H^{1/2,1/4}(\Gamma, \mathbb{R}_+)$.

In section 3, we define spaces of functions over $\mathbb{R}^3 \times \mathbb{R}$ which are analogous to the space $W^1(\mathbb{R}^3)$ and establish the mapping properties of the heat operator on them. In section 4 we prove a major result of this paper. We show that the single layer heat potential operator $\mathcal{K}_1$ extends from smooth functions to an isomorphism of $H^{-1/2,-1/4}(\Gamma, \mathbb{R}_+)$ isomorphically onto its dual $H^{1/2,1/4}(\Gamma, \mathbb{R}_+)$. Furthermore, we show that $\mathcal{K}_1$ satisfies the coercivity estimate

$$\langle q, \mathcal{K}_1 q\rangle \geq c\|q\|^2_{H^{-1/2,-1/4}(\Gamma,\mathbb{R}_+)}.$$

It is remarkable that the single layer heat potential satisfies such a coercivity estimate. This kind of estimate is more typical of elliptic operators and it does not hold for the heat operator. Besides being of theoretical interest, this coercivity has immediate applications to the study of Galerkin discretization methods which are known to be quasioptimal when applied to coercive operators.

Before discussing the Galerkin methods, we consider the regularity of the mapping $\mathcal{K}_1$. Letting ($s \in \mathbb{R}$),

$$\begin{aligned} X^s_{00}(\Gamma, \mathbb{R}_+) \;\; &= \;\; \{u \in H^{2s,s}(\Gamma, \mathbb{R}_+) \colon \exists U \in H^{2s,s} \text{ such that} \\ &\qquad U = u \text{ (a.e) } t > 0,\, U = 0 \text{ (a.e) } t < 0\}, \end{aligned}$$

we show that $\mathcal{K}_1$: $X_{00}^{r/2-1/4}(\Gamma, \mathbb{R}_+) \to X_{00}^{r/2+1/4}(\Gamma, \mathbb{R}_+)$ is an isomorphism for all non-negative r. The case $r = 1/2$ of these results agrees with the results of R. Brown [7] who showed (by very different methods) that $\mathcal{K}_1$ is an isomorphism of $L^2(\Gamma \times \mathbb{R}_+)$ onto $H^{1,1/2}(\Gamma, \mathbb{R}_+)$ for any Lipschitz surface Γ. This discussion is contained in section 5.

In section 6 and section 7, we discuss the implementation and error analysis for a Galerkin discretization of (1.11). We study the Galerkin method since a complete error analysis for the method may be given. The error analysis shows that if the different mesh sizes in time and space concurrently decrease in a appropriate way, the Galerkin method converges with optimal order in $L^2(\Gamma \times \mathbb{R}_+)$. To test this and other claims, we present some numerical examples in section 8.

2 The Anisotropic Sobolev Spaces: Trace Theory

In this section, we review the definition and theory of the anisotropic Sobolev spaces. In section 2.1, we recall the definitions of the spaces over $\mathbb{R}^3 \times \mathbb{R}$. The remaining sections focus on the mapping properties of the trace operator.

2.1 Preliminaries and the Time Restriction Operator

As noted in the introduction, the anisotropic Sobolev spaces $H^{r,s}$ are defined for all $r, s \geq 0$ by

$$H^{r,s} = L^2(\mathbb{R}, H^r(\mathbb{R}^3)) \cap H^s(\mathbb{R}, L^2(\mathbb{R}^3)).$$

Norms over $H^{r,s}$ may be defined using Fourier transforms in space and time. We will denote this operator by $\mathcal{F}_{x,t}$ and will assume that it is defined on smooth functions $u \in \mathcal{S}(\mathbb{R}^3 \times \mathbb{R})$ by

$$\mathcal{F}_{x,t}(u) = \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} u(x,t) e^{-ix\xi} e^{-it\tau} dx dt, \quad (\xi, \tau) \in \mathbb{R}^3 \times \mathbb{R},$$

and extended to $\mathcal{S}'(\mathbb{R}^3 \times \mathbb{R})$ by Parseval's theorem. In terms of the Fourier transform, an equivalent norm on $H^{r,s}$ is given by

$$\|u\|^2_{H^{r,s}} = \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} \left[(1 + |\xi|^2)^r + (1 + |\tau|^2)^s \right] |\mathcal{F}_{x,t}(u)(\xi, \tau)|^2 d\xi d\tau. \qquad (2.1)$$

The continuous inclusions

$$\mathcal{D}(\mathbb{R}^3 \times \mathbb{R}) \subset L^2(\mathbb{R}^3 \times \mathbb{R})) \subset H^{r_1,s_1} \subset H^{r_2,s_2}, \quad 0 \leq r_1 \leq r_2, \, 0 \leq s_1 \leq s_2,$$

are dense. Once and for all, we note the existence of positive constants $c_{r,s}$ and $C_{r,s}$ such that

$$c_{r,s}(1 + |\xi|^{2r} + |\tau|^{2s}) \leq \left\{ (1 + |\xi|^2)^r + (1 + |\tau|^2)^s \right\} \leq C_{r,s}(1 + |\xi|^{2r} + |\tau|^{2s}),$$

for all $r, s \geq 0$.

In this paper, it will be convenient to consider the spaces $H^{r,s}$ as being complex valued. Thus, the appropriate definition of the spaces $H^{-r,-s}$ is as the antidual space to $H^{r,s}$. That is, the space of continuous, antilinear forms. (An antilinear form means a mapping f: $H^{r,s} \to \mathbb{C}$ which satisfies

$$f(\lambda_1 u_1 + \lambda_2 u_2) = \overline{\lambda_1} f(u_1) + \overline{\lambda_2} f(u_2), \quad u_1, u_2 \in H^{r,s}, \lambda_1, \lambda_2 \in \mathbb{C}.)$$

It is not hard to show that the spaces $H^{-r,-s}$ so obtained are equivalent to the sum space

$$H^{-r,s} = L^2(\mathbb{R}, H^{-r}(\mathbb{R}^3)) + H^{-s}(\mathbb{R}, L^2(\mathbb{R}^3)),$$

with the sum norm

$$\|f\|^2 = \inf_{f=f_1+f_2} \left(\|f_1\|^2_{L^2(\mathbb{R},H^{-1}(\mathbb{R}^3)} + \|f_2\|^2_{H^{-1/2}(\mathbb{R},L^2(\mathbb{R}^3))} \right), \qquad f \in H^{-r,-s}.$$

They are also equivalent to the set of locally integrable functions f for which

$$f \to \left(\int_{-\infty}^{\infty} \int_{\mathbb{R}^3} \frac{|\mathcal{F}_{x,t}(f)(\xi,\tau)|^2}{(1+|\xi|+|\tau|^{s/r})^{2r}} d\xi d\tau \right)^{1/2},$$

is finite.

With $r \geq 0$ arbitrarily fixed and $s \in (0,1)$, an equivalent norm on $H^{r,s}$ is given by

$$\|u\|^2_{H^{r,s}} = \int_{-\infty}^{\infty} \|u(\cdot,t)\|^2_{H^r(\mathbb{R}^3)} dt + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\|u(\cdot,t) - u(\cdot,t')\|^2_{L^2(\mathbb{R}^3)}}{|t-t'|^{1+2s}} dt dt'. \quad (2.2)$$

For larger values of s, let m equal the integer part of s and set $\sigma = s - m$. Then, the expression

$$\|u\|^2_{H^{r,s}} = \|u\|^2_{H^{r,\sigma}} + \|u^{(m)}\|^2_{H^{r,\sigma}}, \qquad (2.3)$$

defines an equivalent norm on $H^{r,s}$. Based on (2.2–2.3), a natural definition of the spaces $H^{r,s}(\mathbb{R}^3, \mathbb{R}_+)$ for $s \in (0,1)$ is made using the norm

$$\begin{aligned} \|u\|^2_{H^{r,s}(\mathbb{R}^3,\mathbb{R}_+)} &= \int_0^{\infty} \|u(\cdot,t)\|^2_{H^r(\mathbb{R}^3)} dt \\ &\quad + \int_0^{\infty} \int_0^{\infty} \frac{\|u^{(m)}(\cdot,t) - u^{(m)}(\cdot,t')\|^2_{L^2(\mathbb{R}^3)}}{|t-t'|^{1+2\sigma}} dt dt'. \end{aligned}$$

Similarly, for higher values of s, one can define the spaces $H^{r,s}(\mathbb{R}^3,\mathbb{R})$ by the norms

$$\|u\|^2_{H^{r,s}(\mathbb{R}^3,\mathbb{R}_+)} = \|u\|^2_{H^{r,\sigma}(\mathbb{R}^3,\mathbb{R}_+)} + \|u^{(m)}\|^2_{H^{r,\sigma}(\mathbb{R}^3,\mathbb{R})}.$$

Using (2.2–2.3), it is simple to check that the restriction of $U \in H^{r,s}$ to the set $\mathbb{R}^3 \times \mathbb{R}_+$ belongs to $H^{r,s}(\mathbb{R}^3,\mathbb{R}_+)$ for any $r, s > 0$. It is well known that this restriction operator actually maps $H^{r,s}(\mathbb{R}^3,\mathbb{R})$ onto $H^{r,s}(\mathbb{R}^3,\mathbb{R}_+)$ for all $r, s \geq 0$. This follows from the existence of an extension operator which simultaneously extends $H^{r,s}(\mathbb{R}^3,\mathbb{R}_+)$ to $H^{r,s}$ for all $r, s \geq 0$. For many purposes, such as ours, this weaker result suffices.

Lemma 2.1 *For each positive integer M, there exists an extension operator E_M which depends on M such that*

$$E_M \colon H^{r,s}(\mathbb{R}^3,\mathbb{R}_+) \to H^{r,s} \quad \textit{for all} \quad s \in [0, M], \ \textit{any}\ r \geq 0,$$

with $E_M u(x,t) = u(x,t)$ for almost all $(x,t) \in \mathbb{R}^3 \times \mathbb{R}_+$.

Remark: A standard choice for E_M is the operator

$$E_M u(x,t) = \begin{cases} u(x,t), & x \in \mathbb{R}^3,\ t > 0, \\ \sum_{j=0}^m \alpha_j u(x,-jt), & x \in \mathbb{R}^3,\ t < 0, \end{cases}$$

where the coefficients α_j are chosen to satisfy

$$\sum_{j=1}^m (-j)^k \alpha_j = 1 \qquad 0 \leq k \leq m-1.$$

There are two important subspaces of $H^{r,s}(\mathbb{R}^3,\mathbb{R}_+)$ we will need to use. One is the subspace $H_0^{r,s}(\mathbb{R}^3,\mathbb{R}_+)$ which is defined as the closure of $\mathcal{D}(\mathbb{R}^3,\mathbb{R}_+)$ in the $H^{r,s}(\mathbb{R}^3,\mathbb{R}_+)$ norm. For all $s > 1/2$, this is a strictly proper subspace of $H^{r,s}(\mathbb{R}^3,\mathbb{R}_+)$. For $s \in [0, 1/2]$, however, this space coincides with $H^{r,s}(\mathbb{R}^3,\mathbb{R}_+)$. (The case $s = 1/2$ is a non-trivial result which is proven in [24].)

The other subspace we need is the space $H_{00}^{r,s}(\mathbb{R}^3,\mathbb{R}_+)$. This space is defined to be the space of functions u such that there exists some function $U \in H^{r,s}(\mathbb{R}^3,\mathbb{R})$ which agrees with u for all $t > 0$ and vanishes for all $t < 0$. It is a Hilbert space when given the norm

$$\|u\|_{H^{r,s}(\mathbb{R}^3,\mathbb{R}_+)} = \inf_U \|U\|_{H^{r,s}},$$

where the infimum is taken over all such U's. Although the definition of this space is different than the definition of the space $H_0^{r,s}(\mathbb{R}^3,\mathbb{R}_+)$, it turns out that these two spaces coincide for all $r,s \geq 0$ except for the s values which satisfy $s = m + 1/2$ with $m \in \mathbb{N}$. For these special values of s, the space $H_{00}^{r,m+1/2}(\mathbb{R}^3,\mathbb{R}_+)$ is a strictly proper subspace of $H_0^{r,m+1/2}(\mathbb{R}^3,\mathbb{R}_+)$.

Remark: It is customary to define the spaces $H_{00}^{r,s}(\mathbb{R}^3,\mathbb{R}_+)$ only for $s - 1/2 \in \mathbb{N}$, since they agree with $H_0^{r,s}(\mathbb{R}^3,\mathbb{R}_+)$ otherwise. In this paper, however, it is natural to define $H_{00}^{r,s}$ for all $r,s \geq 0$ since they are the proper setting for the regularity theory in section 5.

The negative indexed Sobolev spaces $H^{-r,-s}(\mathbb{R}^3,\mathbb{R}_+)$ are defined as the antidual spaces to $H_0^{r,s}(\mathbb{R}^3,\mathbb{R}_+)$ with corresponding norm

$$\|q\|_{H^{-r,-s}(\mathbb{R}^3,\mathbb{R}_+)} = \sup_{g \in H_0^{r,s}(\Gamma,\mathbb{R}_+)} \frac{\langle q, g\rangle}{\|g\|_{H^{r,s}(\Gamma,\mathbb{R}_+)}}.$$

Again, these spaces can be shown to be equivalent to the sum spaces

$$H^{-r,s} = L^2(\mathbb{R}_+, H^{-r}(\mathbb{R}^3)) + H^{-s}(\mathbb{R}_+, L^2(\mathbb{R}^3)),$$

with the sum norm

$$\|f\|^2 = \inf_{f=f_1+f_2} \left(\|f_1\|^2_{L^2(\mathbb{R}_+,H^{-r}(\mathbb{R}^3))} + \|f_2\|^2_{H^{-s}(\mathbb{R}_+,L^2(\mathbb{R}^3))} \right).$$

It is important to note that $H^{-r,-s}(\Gamma,\mathbb{R}_+)$ is not the antidual space of $H^{r,s}(\mathbb{R}^3,\mathbb{R}_+)$ for all $s \geq 1/2$.

Clearly, the spaces $H^{r,s}(\mathbb{R}^3,\mathbb{R}_+)$ and $H_{00}^{r,s}(\mathbb{R}^3,\mathbb{R}_+)$ are intimately connected with the time restriction operator R_+ and the zero extension operator Z_+. (That is,

$$Z_+u(x,t) = \left\{ \begin{array}{ll} u(x,t), & x \in \mathbb{R}^3,\, t > 0, \\ 0, & x \in \mathbb{R}^3,\, t < 0. \end{array} \right)$$

By definition, each of the mappings R_+: $H^{r,s} \to H^{r,s}(\mathbb{R}^3,\mathbb{R}_+)$ and Z_+: $H_{00}^{r,s}(\Gamma,\mathbb{R}_+) \to H^{r,s}$ are bounded for all $r,s \geq 0$. Both of these operators can be extended into the dual spaces. To extend Z_+, consider the adjoint mapping R_+^*: $(H^{r,s}(\mathbb{R}^3,\mathbb{R}_+))^* \to H^{-r,-s}$. For all $f \in H^{-r,-s}$, this map is defined by

$$\langle R_+^* f, U\rangle = \langle f, R_+^* U\rangle \quad \text{for all} \quad U \in H^{r,s}.$$

Thus, for $f \in L^2(\mathbb{R}^3 \times \mathbb{R})$, we have

$$\langle R_+^* f, U\rangle = \int_0^\infty \int_{\mathbb{R}^3} f(x,t)\overline{U(x,t)}dxdt, \quad \text{for all} \quad U \in H^{r,s},$$

which agrees with the zero extension of f. Analogously, the adjoint operator Z_+^* extends the restriction operator R_+. The equality of $H_0^{r,s}(\mathbb{R}^3, \mathbb{R}_+)$ with $H_{00}^{r,s}(\mathbb{R}^3, \mathbb{R}_+)$ for all $r, s \geq 0$ such that $s - 1/2 \notin \mathbb{N}$ shows that R_+: $H^{-r,-s} \to H^{r,s}(\mathbb{R}^3, \mathbb{R}_+)$ is bounded for these values. For easy reference, we summarize the mapping properties of these operators in a lemma.

Lemma 2.2 *Let R_+: $\mathcal{D}(\mathbb{R}^3 \times \mathbb{R}) \to C^\infty(\mathbb{R}^3 \times \mathbb{R}_+)$ denote the restriction operator in time. Then, R_+ extends to a bounded linear mappings of*

$$H^{r,s}(\mathbb{R}^3, \mathbb{R}) \to H^{r,s}(\mathbb{R}^3, \mathbb{R}_+) \quad \textit{for all} \quad r, s \geq 0,$$
$$H^{-r,-s}(\mathbb{R}^3, \mathbb{R}) \to H^{-r,-s}(\mathbb{R}^3, \mathbb{R}_+) \quad \textit{for all} \quad r, s \geq 0, \textit{ such that } s - 1/2 \notin \mathbb{N}.$$

Analogously, let Z_+: $\mathcal{D}(\mathbb{R}^3 \times \mathbb{R}_+) \to \mathcal{D}(\mathbb{R}^3 \times \mathbb{R})$ denote the operator of extension by zero to $t < 0$. Then, Z_+ extends to a bounded linear mapping of

$$H_{00}^{r,s}(\mathbb{R}^3, \mathbb{R}_+) \to H^{r,s}, \quad \textit{for all} \quad r, s \geq 0.$$

We will also use anisotropic Sobolev spaces defined over other spatial regions besides $\mathbb{R}^3$. For any $r, s \geq 0$ and any open set $O \in \mathbb{R}^3$, the spaces $H^{r,s}(O, \mathbb{R})$ can be defined as the space of restrictions to $O \times \mathbb{R}$ of $H^{r,s}$. The corresponding norm of this space is

$$\|u\|_{H^{r,s}(O,\mathbb{R})} = \inf \|U\|_{H^{r,s}},$$

where the infimum is taken over all $U \in H^{r,s}$ which agree with u on $O \times \mathbb{R}$. Equivalently, these are the spaces

$$H^{r,s}(O, \mathbb{R}) = L^2(\mathbb{R}, H^r(O)) \cap H^s(\mathbb{R}, L^2(O)).$$

In this paper, the set O shall either be a bounded set in $\mathbb{R}^3$, which we shall denote by Ω, or the complement of such a set, which we will denote by Ω^c. In addition to $H^{r,s}(O, \mathbb{R})$, we introduce the space $H_0^{r,s}(O \times \mathbb{R})$ which is defined as the closure of $\mathcal{D}(O \times \mathbb{R})$ in the $H^{r,s}(O, \mathbb{R})$ norm. Many properties are known about both of these spaces. For example, we have

$$H_0^{r,s}(O \times \mathbb{R}) = H^{r,s}(O \times \mathbb{R}), \qquad 0 \leq r, s < 1/2,$$

with $H_0^{r,s}(\Omega \times \mathbb{R})$ strictly included in $H^{r,s}(O \times \mathbb{R})$ otherwise. A complete discussion of these spaces can be found in [25, Chapter 4]. For our purposes, it will be sufficient to simply state what else we need at the appropriate time.

2.2 Definition of the Trace Operator

In the next two sections, we will discuss the theory of the trace operator on the anisotropic Sobolev spaces. The trace operator considers an extension of the restriction operator from functions on $\mathbb{R}^3 \times \mathbb{R}$ to functions on $\Gamma \times \mathbb{R}$. Although this operator is well understood [25, Chapter 4], we will present its theory in detail. Mostly, this is because we need to develop some facts which are not discussed in [25]. In this section, we describe the way in which the trace operator is defined through regularization and localization.

To simplify the exposition, we assume throughout that Ω refers to a bounded, open set in $\mathbb{R}^3$ whose boundary Γ is an infinitely differentiable manifold of dimension two and that Ω lies to one side of Γ. This stringent assumption on the smoothness of Γ allows us to discuss the trace mapping on the spaces $H^{r,s}(\mathbb{R}^3, \mathbb{R})$ for all positive values of r. The weaker assumption that Γ is a C^{k+1} surface for integer $k \geq 1$ would suffice to discuss the trace mapping for all $r \leq k$. Even this assumption on Γ could be weakened, but we will not discuss this matter here. Throughout this section, r and s denote non-negative real numbers with $r > 1/2$. We set

$$\lambda = r - 1/2 \quad \text{and} \quad \mu = \frac{s}{r}(r - 1/2).$$

The smoothness assumptions on Γ imply that there exists a finite covering of $\bar{\Omega}$ by bounded, open sets O_1 thru O_M such that for each integer j between 1 and M, there is a C^∞ diffeomorphisms ϕ_j mapping

$$\begin{aligned} O_j &\quad \text{onto} \quad Y = \{(y_1, y_2, y_3) \colon |y_i| < 1,\ i = 1, 2, 3\}, \\ O_j \cap \Omega &\quad \text{onto} \quad Y_+ = \{(y_1, y_2, y_3) \colon |y_i| < 1,\ i = 1, 2,\ 0 < y_3 < 1\}, \\ O_j \cap \Gamma &\quad \text{onto} \quad Y_0 = \{y \in Y : y_3 = 0\}. \end{aligned}$$

Furthermore, there exists an open subset O_0 with closure contained in Ω such that the sets $O_0, O_1, \ldots, O_M$ cover of $\bar{\Omega}$. For each j, we let ψ_j equal the inverse mappings of ϕ_j and for notational convenience introduce the operators

$$\psi_j^*(w)(x,t) = w(\phi_j(x), t) \qquad w \in H^{r,s}(Y, \mathbb{R}), \tag{2.4}$$

$$\phi_j^*(u)(y,t) = u(\psi_j(y), t), \qquad u \in H^{r,s}(O_j, \mathbb{R}). \tag{2.5}$$

We have

$$\|\psi_j^*(w)\|_{H^{r,s}(O_j,\mathbb{R})} \leq C(\Omega)\|w\|_{H^{r,s}(Y,\mathbb{R})}, \tag{2.6}$$

$$\|\phi_j^*(u)\|_{H^{r,s}(Y,\mathbb{R})} \leq C(\Omega)\|u\|_{H^{r,s}(O_j,\mathbb{R})}. \tag{2.7}$$

We now introduce a partition of unity subordinate to this covering of Ω. That is, for each j between 0 and M, we let $\zeta_j(x) \in \mathcal{D}(\mathbb{R}^3)$ denote a non-negative function which is supported in O_j such that

$$\sum_{j=0}^{M} \zeta_j(x) = 1, \qquad x \in \Omega.$$

Details on the construction of these functions may be found in [16, pp. 19-20]. Since we will need it shortly, we point out here that we can assume without loss of generality that the square root $\zeta_j^{1/2}$ of each of these functions also belongs to $\mathcal{D}(O_j)$. (If not, we define a new class of functions as the squares of the original ones and then normalize them.) We then set

$$\zeta_{M+1}(x) = 1 - \sum_{j=0}^{M} \zeta_j(x), \qquad x \in \mathbb{R}^3,$$

to arrive at a partition of unity $\{\zeta_j\}_{j=0}^{M+1}$ of $\mathbb{R}^3$. Using these functions, we can write any $u \in H^{r,s}(\mathbb{R}^3, \mathbb{R})$ as

$$u(x,t) = \sum_{j=0}^{M+1} \zeta_j(x) u(x,t), \qquad (x,t) \in \mathbb{R}^3 \times \mathbb{R}. \tag{2.8}$$

Note that the x support of $\zeta_{M+1}(x)u(x,t)$ is disjoint from Ω, while the x support of $\zeta_0(x)u(x,t)$ has closure contained in Ω. Thus, these two functions vanish in a x neighborhood of Γ. Introducing the maps ψ_j^* and ϕ_j^* into (2.8) leads to the equality

$$\begin{aligned} u(x,t) &= \zeta_0(x)u(x,t) + \sum_{j=1}^{M} \psi_j^* \left(\phi_j^*(\zeta_j u)\right)(x,t) \\ &\quad + \zeta_{M+1}(x)u(x,t), \qquad (x,t) \in \mathbb{R}^3 \times \mathbb{R}. \end{aligned} \tag{2.9}$$

Equation (2.9) is the basis of localization. To formalize this process, let W denote the product Hilbert space

$$W = H_0^{r,s}(\Omega, \mathbb{R}) \times \prod_{j=1}^{M} H_0^r(Y, \mathbb{R}) \times H_0^{r,s}(\Omega^c, \mathbb{R}),$$

with associated norm

$$\begin{aligned}\|\vec{w}\|_W^2 &= \|w_0\|^2_{H^{r,s}(\Omega,\mathbb{R})} + \sum_{j=1}^{M} \|w_j\|^2_{H^{r,s}(Y,\mathbb{R})} \\ &\quad + \|w_{M+1}\|^2_{H^{r,s}(\Omega^c,\mathbb{R})}, \quad (w_0, w_1, \ldots, w_{M+1}) \in W.\end{aligned}$$

Now, define T: $H^{r,s}(\mathbb{R}^3,\mathbb{R})$ to W by

$$Tu = (\zeta_0 u,\, \phi_1^*(\zeta_1 u), \ldots, \phi_M^*(\zeta_M u),\, \zeta_{M+1} u)\,. \tag{2.10}$$

Clearly, T maps $H^{r,s}(\mathbb{R}^3,\mathbb{R})$ boundedly into W. A left inverse to T is given by the mapping

$$\begin{aligned}\nu(w_0, w_1, \ldots, w_{M+1}) &= w_0(x,t) + \sum_{j=1}^{M} \psi_j^*(w_j)(x,t) \\ &\quad + w_{M+1}(x,t), \qquad (x,t) \in \mathbb{R}^3 \times \mathbb{R}.\end{aligned}$$

Since

$$\|u\|_{H^{r,s}(\mathbb{R}^3,\mathbb{R})} = \|\nu T u\|_{H^{r,s}(\mathbb{R}^3,\mathbb{R})} \leq C(\Omega)\|Tu\|_W, \tag{2.11}$$

T defines an isomorphism between $H^{r,s}(\mathbb{R}^3,\mathbb{R})$ and its range $\mathcal{R}(T)$ which is a closed subspace of W. Note that ν is not a right inverse of T.

Of course, before the trace map can be defined, we must define the Sobolev spaces $H^{r,s}(\Gamma,\mathbb{R})$. For each integer j between 1 and M, we let $\mathcal{O}_j = O_j \cap \Gamma$, and set

$$\breve{\zeta}_j(x) = \zeta_j(x), \qquad x \in \mathcal{O}_j.$$

Note that $\mathcal{O}_1$, $\mathcal{O}_2$, …, $\mathcal{O}_M$ cover Γ and that $\breve{\zeta}_1$, …, $\breve{\zeta}_M$ are a partition of unity subordinate to this cover. Again, we introduce the operators

$$\begin{aligned}\psi_j^*\,(g)\,(x,t) &= g(\phi_j(x),t), \qquad g \in \mathcal{D}(Y_0 \times \mathbb{R}), \\ \phi_j^*\,(v)\,(y,t) &= v(\psi_j(y),t), \qquad v \in \mathcal{D}(\mathcal{O}_j \times \mathbb{R}).\end{aligned}$$

The product mapping T' defined by

$$T'v = \left(\phi_1^*(\breve{\zeta}_1 v), \ldots, (\phi_M^*(\breve{\zeta}_M v)\right)$$

embeds smooth functions v into a product space, say G, of functions which are defined over $\mathbb{R}^2 \times \mathbb{R}$. The mapping ν' given by

$$\nu'(g_1, g_2, \cdots, g_M) = \sum_{j=1}^{M} \psi_j^*\,(g_j)\,(x,t), \qquad (x,t) \in \Gamma \times \mathbb{R},$$

is a left inverse of T'. The norm

$$\|v\|^2_{H^{r,s}(\Gamma,\mathbb{R})} = \sum_{j=1}^{M} \|\phi_j^* \left(\breve{\zeta}_j v\right) \|^2_{H^{r,s}(\mathbb{R}^2,\mathbb{R})},$$

induces a Hilbert space structure on functions defined over $\Gamma \times \mathbb{R}$. It is well known (see [44, Chapter 25]) that all choices of covering sets and partitions of unity lead to equivalent norms.

Let R_3 denote the operator of restriction from functions defined on $(y', y_3, t) \in \mathbb{R}^3 \times \mathbb{R}$ to ones defined on $(y', t) \in \mathbb{R}^2 \times \mathbb{R}$. We also view R_3 as an operator from W to G by

$$R_3(w_0, w_1, w_2, \dots, w_{M+1}) = (R_3 w_1, R_3 w_2, \dots, R_3 w_M).$$

The trace operator γ is then defined for $u \in H^{r,s}(\mathbb{R}^3, \mathbb{R})$ by

$$\gamma u = \nu' R_3 T u.$$

By construction, γ agrees with the restriction operator when applied to smooth functions. Given that the operator R_3 maps $H^{r,s}(\mathbb{R}^3, \mathbb{R})$ into $H^{\lambda,\mu}(\mathbb{R}^2, \mathbb{R})$, it follows by definition that γ maps $H^{r,s}(\mathbb{R}^3, \mathbb{R})$ into $H^{\lambda,\mu}(\Gamma, \mathbb{R})$. Similarly, if R_3 maps $H^{r,s}(\mathbb{R}^3, \mathbb{R})$ onto $H^{\lambda,\mu}(\mathbb{R}^2, \mathbb{R})$, then γ maps $H^{r,s}(\mathbb{R}^3, \mathbb{R})$ onto $H^{\lambda,\mu}(\Gamma, \mathbb{R})$. We elaborate on this and show how to extend a given $v \in H^{\lambda,\mu}(\Gamma, \mathbb{R}) \to u \in H^{r,s}(\mathbb{R}^3, \mathbb{R})$ such that $\gamma u = v$.

Let $v \in H^{\lambda,\mu}(\Gamma, \mathbb{R})$ be given and assume that E is a right inverse to R_3. Then, by definition, the function $\phi_m^* \left(\breve{\zeta}_m v\right)(y, t)$ belongs to $H^{\lambda,\mu}(\mathbb{R}^2, \mathbb{R})$ for each integer m between 1 and M. Using the extension operator E, consider the following function in $H^{r,s}(\mathbb{R}^3, \mathbb{R})$:

$$u_m(x, t) = \psi_j^* \left[\phi_m^*(\zeta_m^{1/2}) E \left(\phi_m^* \left(\breve{\zeta}_m^{1/2} v\right)\right)\right](x, t), \qquad (x, t) \in \mathbb{R}^3 \times \mathbb{R}. \quad (2.12)$$

Expanding this definition out, one can check that $u_m \in H^{r,s}(\mathbb{R}^3, \mathbb{R})$ and has x support in $O_m \times \mathbb{R}$. Its trace on Γ is

$$\gamma u_m(x, t) = \breve{\zeta}_m(x)(x) v(x, t), \qquad (x, t) \in \Gamma \times \mathbb{R}.$$

Using the linearity of the trace operator and the fact that the functions $\breve{\zeta}_m(x)$ are a partition of unity with respect to Γ, it follows that the function

$$u(x, t) = \sum_{m=1}^{M} u_m(x, t), \qquad (x, t) \in \mathbb{R}^3 \times \mathbb{R}$$

belongs to $H^{r,s}(\mathbb{R}^3, \mathbb{R})$ and satisfies $\gamma u = v$.

Remark: Higher order trace operators $\gamma^{(k)}$ for integer values of k are defined by

$$\gamma^{(k)} u = \nu' R_3 \frac{\partial^k}{\partial y_3^k} T u, \qquad k \in \mathbb{Z}_+,$$

where one views $\partial/\partial y_3$ as the operator from W to G defined by

$$\frac{\partial}{\partial y_3}(w_0, \ldots, w_M, w_{M+1}) = (\frac{\partial w_1}{\partial y_3}, \ldots, \frac{\partial w_M}{\partial y_3},).$$

These operators extend the classical definition of the derivatives $\partial^k u/\partial^k \mathbf{n}$ with respect to the surface normal direction $\mathbf{n}$.

2.3 Mapping Properties of the Trace Operator

The discussion in section 2.2 shows how the properties of the trace operator may be reduced to studying the restriction operator R_3 which maps functions $w(y', y_3, t)$ defined on

$$\{y = (y', y_3) \in \mathbb{R}^3 \colon y' \in \mathbb{R}^2,\ y_3 \in \mathbb{R}\},$$

to functions $w(y', 0, t)$ defined on

$$\{y = (y', 0) \in \mathbb{R}^2 \colon y' \in \mathbb{R}^{n-1}\}.$$

Throughout this subsection, we write $y \in \mathbb{R}^3$ as $y = (y', y_3)$ where $y' \in \mathbb{R}^2$ and $y_3 \in \mathbb{R}$. We shall use $\xi = (\xi', \xi_3)$ to denote the corresponding Fourier transform variables to (y', y_3). We set

$$\lambda = r - 1/2 \quad \text{and} \quad \mu = \frac{s}{r}\lambda.$$

Theorem 2.3 *The operator R_3 maps $H^{r,s}(\mathbb{R}^3_+, \mathbb{R})$ onto $H^{\lambda,\mu}(\mathbb{R}_2, \mathbb{R})$ for all $r > 1/2$, $s \geq 0$. Hence, the trace operator γ maps $H^{r,s}(\mathbb{R}^3, \mathbb{R}_+)$ onto $H^{\lambda,\mu}(\Gamma, \mathbb{R}_+)$.*

Proof: It suffices to prove the claim about R_3. Let $\hat{w}(\xi', \xi_3, \tau)$ denote the Fourier transform of w. The basic relationship between w and its traces is given by

$$\mathcal{F}_{y',t}(\gamma w)(\xi', \tau) = \int_{-\infty}^{\infty} \hat{w}(\xi', \xi_3, \tau) d\xi_3. \tag{2.13}$$

Taking absolute values in (2.13) and applying the Cauchy-Schwartz inequality, we get

$$|\mathcal{F}_{y',t}(\gamma w)(\xi',\tau)|^2 \leq \left(\int_{-\infty}^{\infty} \frac{d\xi_3}{k(\xi',\xi_3,\tau)}\right)\left(\int_{-\infty}^{\infty} k(\xi',\xi_3,\tau)|\hat{w}(\xi',\xi_3,\tau)|^2 d\xi_3\right), \tag{2.14}$$

where

$$k(\xi',\xi_3,\tau) = 1 + |\xi'|^{2r} + |\xi_3|^{2r} + |\tau|^{2s}.$$

To bound the first integral of (2.14), we change variables of integration by letting $\xi_3 = (1+|\xi'|^{2r}+|\tau|^{2s})^{1/2r}\sigma$. We find that

$$\int_{-\infty}^{\infty} \frac{d\xi_3}{k(\xi',\xi_3,\tau)} \leq \left(1+|\xi'|^{2r}+|\tau|^{2s}\right)^{-\lambda/r} \int_{-\infty}^{\infty} \frac{d\sigma}{1+\sigma^{2r}}.$$

Since $2r > 1$, the integral over σ is finite and thus (2.14) becomes the inequality

$$\left(1+|\xi'|^{2r}+|\tau|^{2s}\right)^{\lambda/r} |\mathcal{F}_{y',t}(\gamma w)(\xi',\tau)|^2 \leq C\int_{-\infty}^{\infty} k(\xi',\xi_3,\tau)|\hat{w}(\xi',\xi_3,\tau)|^2 d\xi_3.$$

Recalling the definition of μ, it follows by redefining the constant C that

$$\left(1+|\xi'|^{2\lambda}+|\tau|^{2\mu}\right) |\mathcal{F}_{y',t}(\gamma w)(\xi',\tau)|^2 \leq C\int_{-\infty}^{\infty} k(\xi',\xi_3,\tau)|\hat{w}(\xi',\xi_3,\tau)|^2 d\xi_3.$$

Integrating this inequality over ξ' and τ and noting that

$$\int_{-\infty}^{\infty}\int_{\mathbb{R}^2}\int_{-\infty}^{\infty} k(\xi',\xi_3,\tau)|\hat{w}(\xi',\xi_3,\tau)|^2 d\xi_3 d\xi' d\tau \leq C(r,s)\|w\|^2_{H^{r,s}}, \quad w \in H^{r,s},$$

we get

$$\|\gamma w\|_{H^{\lambda,\mu}(\mathbb{R}^2,\mathbb{R})} \leq C(r,s)\|w\|_{H^{r,s}(\mathbb{R}^2,\mathbb{R})}. \tag{2.15}$$

Equation (2.15) shows that γ extends to bounded, linear mapping of $H^{r,s}(\mathbb{R}^3,\mathbb{R})$ into $H^{\lambda,\mu}(\mathbb{R}^2,\mathbb{R})$. To show that it maps onto $H^{\lambda,\mu}(\mathbb{R}^2,\mathbb{R})$, we construct an extension mapping. Let $\rho(\sigma) \in \mathcal{D}(\mathbb{R})$ be a non-negative function which satisfies

$$\int_{-\infty}^{\infty} \rho(\sigma) d\sigma = 1.$$

Given $g \in H^{\lambda,\mu}(\mathbb{R}^{n-1}, \mathbb{R})$, let $\hat{g}$ equal its Fourier transform in space and time. Now, define U by

$$\hat{U} := \mathcal{F}_{y',y_3,t}(U)(\xi', \xi_3, \tau) = \frac{\hat{g}(\xi', \tau)}{W(\xi', \tau)} \rho(\xi_3 / W(\xi', \tau)), \tag{2.16}$$

where

$$W(\xi', \tau) = \left(1 + |\xi'| + |\tau|^{s/r}\right).$$

Since

$$1 + |\xi|^{2r} + |\tau|^{2s} \le C_{r,s}\left[(W(\xi,\tau))^{2r} + |\xi_3|^{2r}\right],$$

a simple calculation shows that

$$\int_{-\infty}^{\infty} (1 + |\xi|^{2r} + |\tau|^{2s}) |\hat{U}(\xi, \tau)|^2 d\xi_3 \le C |\hat{g}(\xi', \tau)|^2 (W(\xi', \tau))^{2r-1}. \tag{2.17}$$

Integrating (2.17) over (ξ', τ), we get

$$\|U\|_{H^{r,s}(\mathbb{R}^3,\mathbb{R})} \le C_{r,s} \|g\|_{H^{\lambda,\mu}(\Gamma,\mathbb{R})}.$$

Since

$$\int_{-\infty}^{\infty} U(\xi', \xi_3, \tau) d\xi_3 = \hat{g}(\xi', \tau),$$

it follows from (2.13) that $\gamma U = g$. □

Remark: The proof above extends to consider the higher order trace operators $\gamma^{(k)}$. The mappings $\gamma^{(k)}$: $H^{r,s}(\mathbb{R}^3, \mathbb{R}) \to H^{r-k,s(r-k)/r}(\mathbb{R}^2, \mathbb{R})$ are bounded and surjective for all $s \ge 0$ and $r - k > 1/2$. See [25, Theorem 2.1] for more details.

Recall the Sobolev space

$$V = \{u(x,t) \in L^2(\mathbb{R}, H^1(\mathbb{R}^3)) \colon \frac{\partial u}{\partial t}(x,t) \in L^2(\mathbb{R}, H^{-1}(\mathbb{R}^3))\}.$$

The norm on this space can be given in terms of Fourier transforms by

$$\|u\|_V^2 = \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} \frac{(1 + |\xi|^2)^2 + |\tau|^2}{(1 + |\xi|^2)} |\hat{u}(\xi, \tau)|^2 d\xi d\tau. \tag{2.18}$$

As noted before, we have the dense inclusion

$$V \subset H^{1,1/2}(\mathbb{R}^3, \mathbb{R}).$$

Thus, functions $u(x,t) \in V$ admit a trace γu in the space $H^{1/2,1/4}(\Gamma,\mathbb{R})$. We conclude this section by showing that γ maps V onto $H^{1/2,1/4}(\Gamma,\mathbb{R})$.

We will do this by constructing an operator of extension. We work first in the local coordinates (y', y_3) and show the existence of an extension operator E_3: $H^{1/2,1/4}(\mathbb{R}^2,\mathbb{R}) \to V$ such that

$$E_3 g(y',0,t) = g(y',t), \qquad g \in H^{1/2,1/4}(\mathbb{R}^2,\mathbb{R}).$$

To construct such an operator, we need a lemma.

Lemma 2.4 *Let*

$$k(\xi',\xi_3,\tau) = \frac{1+|\xi|^2}{(1+|\xi|^2)^2+|\tau|^2}.$$

Then, there exists positive constants C_1 and C_2 such that

$$C_1\left(1+|\xi'|^2+|\tau|\right)^{-1/2} \le \int_{-\infty}^{\infty} k(\xi',\xi_3,\tau)d\xi_3 \le C_2\left(1+|\xi'|^2+|\tau|\right)^{-1/2}. \tag{2.19}$$

Proof: By the arithmetic-geometric mean inequality,

$$\frac{1+|\xi|^2}{(1+|\xi|^2+|\tau|)^2} \le k(\xi',\xi_3,\tau) \le 2\frac{1+|\xi|^2}{(1+|\xi|^2+|\tau|)^2}. \tag{2.20}$$

Thus, it suffices to estimate the integral

$$I := \int_{-\infty}^{\infty} \frac{1+|\xi|^2}{(1+|\xi|^2+|\tau|)^2}d\xi_3.$$

We first rewrite I as

$$\begin{aligned} I \quad &= \quad (1+|\xi'|^2)\int_{-\infty}^{\infty} \frac{d\xi_3}{(1+|\xi|^2+|\tau|)^2} \\ &\quad + \int_{-\infty}^{\infty} \frac{|\xi_3|^2}{(1+|\xi|^2+|\tau|)^2}d\xi_3. \end{aligned}$$

In each integral, let

$$\xi_3 = \left(1+|\xi'|^2+|\tau|\right)^{1/2}\sigma.$$

This leads to

$$I = \frac{1+|\xi'|^2}{(1+|\xi'|^2+|\tau|)^{3/2}} \int_{-\infty}^{\infty} \frac{d\sigma}{(1+\sigma^2)^2} + \left(1+|\xi'|^2+|\tau|\right)^{-1/2} \int_{-\infty}^{\infty} \frac{\sigma^2}{(1+\sigma^2)^2} d\sigma. \tag{2.21}$$

Setting

$$C_1 = \int_{-\infty}^{\infty} \frac{d\sigma}{(1+\sigma^2)^2} \quad \text{and} \quad C_2 = \int_{-\infty}^{\infty} \frac{\sigma^2}{(1+\sigma^2)^2} d\sigma,$$

(2.21) shows that

$$C_2 \left(1+|\xi'|^2+|\tau|\right)^{-1/2} \le I \le (C_1+C_2)\left(1+|\xi'|^2+|\tau|\right)^{-1/2},$$

as desired. □

We now construct the operator E_3. Let

$$\check{k}(\xi',\tau) = \int_{-\infty}^{\infty} k(\xi',\xi_3,\tau)d\xi_3.$$

By lemma 2.4, we have

$$C_1 \le \check{k}(\xi',\tau)(1+|\xi'|^2+|\tau|)^{1/2} \le C_2.$$

For any $g \in H^{1/2,1/4}(\mathbb{R}^2,\mathbb{R})$, we let

$$\mathcal{F}_{y',y_3,t}\left(E_3 g\right)(\xi',\xi,\tau) = \frac{k(\xi',\xi_3,\tau)}{\check{k}(\xi',\tau)} \mathcal{F}_{y',t}\left(g\right)(\xi',\tau).$$

By construction, we have

$$\int_{-\infty}^{\infty} \mathcal{F}_{y',y_3,t}\left(E_3 g\right)(\xi',\xi_3,\tau)d\xi_3 = \mathcal{F}_{y',t}\left(g\right)(\xi',\tau),$$

and thus $\gamma E_3 g = g$. To show that $E_3 g \in V$, we note that

$$\frac{|\mathcal{F}_{y',y_3,t}\left(E_3 g\right)(\xi',\xi_3,\tau)|^2}{k(\xi',\xi_3,\tau)} = \frac{k(\xi',\xi_3,\tau)}{\check{k}^2(\xi',\tau)} |\mathcal{F}_{y',t}\left(g\right)(\xi',\tau)|^2 d\xi_3.$$

Thus, integrating both sides over ξ_3 and using the definition of $\check{k}(\xi',\tau)$, we get

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{|\mathcal{F}_{y',y_3,t}\left(E_3 g\right)(\xi',\xi_3,\tau)|^2}{k(\xi',\xi_3,\tau)}d\xi_3 &= \int_{-\infty}^{\infty}\frac{k(\xi',\xi_3,\tau)}{\check{k}^2(\xi',\tau)}|\mathcal{F}_{y',t}\left(g\right)(\xi',\tau)|^2 d\xi_3 \\ &= \frac{|\mathcal{F}_{y',t}\left(g\right)(\xi',\tau)|^2}{\check{k}(\xi',\tau)} \\ &\leq \frac{1}{C_1}(1+|\xi'|^2+|\tau|)^{1/2}|\mathcal{F}_{y',t}\left(g\right)(\xi',\tau)|^2.\end{aligned}$$

Integrating this equation over ξ' and τ immediately leads to

$$\begin{aligned}\|E_3 g\|_V^2 &= \int_{-\infty}^{\infty}\int_{\mathbb{R}^2}\int_{-\infty}^{\infty}\frac{|\hat{u}(\xi,\tau)|^2}{k(\xi',\xi_3,\tau)}d\xi_3 d\xi' d\tau \\ &\leq C\|g\|^2_{H^{1/2,1/4}(\mathbb{R}^2,\mathbb{R})}. \end{aligned} \tag{2.22}$$

We now use the operator E_3 to prove our claim on general surfaces Γ. Let $v \in H^{1/2,1/4}(\Gamma,\mathbb{R})$ be given and consider the function

$$u_m(x,t) = \psi_m^*\left[\phi_m^*(\zeta_m^{1/2})E_3(\phi_m^*\left(\zeta_m^{1/2}v\right))\right], \qquad (x,t)\in\mathbb{R}^3\times\mathbb{R}.$$

To show that $u_m \in V$, we compute its $L^2(\mathbb{R}^3\times\mathbb{R})$ inner product with any $\theta\in\mathcal{D}(\mathbb{R}^3\times\mathbb{R})$. We have

$$\begin{aligned}\langle u_m,\theta\rangle &= \int_{-\infty}^{\infty}\int_{\mathbb{R}^3}u_m(x,t)\theta(x,t)dxdt \qquad (2.23)\\ &= \int_{-\infty}^{\infty}\int_{Y}\zeta_m^{1/2}(\psi_m(y))E_3(\phi_m^*(\zeta_m^{1/2}v))J\left(\psi_m\right)(y)\theta(\psi_m(y),t)dydt,\end{aligned}$$

where $J(\psi_m)(y)$ denotes the Jacobian of the transformation from O_m to Y.

Now, the mapping of $\mathcal{D}(\mathbb{R}^3\times\mathbb{R})$ defined by

$$\theta(x,t)\mapsto\zeta_m^{1/2}(\psi_m(y))J\left(\psi_m\right)(y)\theta(\psi_m(y),t),$$

clearly defines a function of (y,t) which belongs to $\mathcal{D}(Y\times\mathbb{R})$. We set $\Psi_m(\theta)$ equal to this map and let $\tilde{\Psi}_m(\theta)$ denote the mapping obtained by extending $\Psi_m\left(\theta\right)(y,t)$ by zero to all $y\in\mathbb{R}^3$. We note a few facts. First, we have the estimate

$$\|\tilde{\Psi}_m\left(\theta\right)\|_{L^2(\mathbb{R},H^1(\mathbb{R}^3))}\leq C(\Omega)\|\theta\|_{L^2(\mathbb{R},H^1(\mathbb{R}^3))}. \tag{2.24}$$

Secondly, by (2.23), we have

$$\langle u_m, \theta \rangle = \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} E_3 \left(\phi_m^*(\zeta_m^{1/2} v) \right) \tilde{\Psi}_m (\theta) (y,t) dy dt. \tag{2.25}$$

Lastly, and most importantly, the mappings $\tilde{\Psi}_m$ commute with the time derivative. That is,

$$\tilde{\Psi}_m \left(\frac{\partial \theta}{\partial t} \right) = \frac{\partial}{\partial t} \tilde{\Psi}_m (\theta), \qquad \theta \in \mathcal{D}(\mathbb{R}^3 \times \mathbb{R}). \tag{2.26}$$

for each m.

From the preceding considerations, we have these expressions for the distributional time derivative of $u_m(x,t)$:

$$\begin{aligned} \langle \frac{\partial u_m}{\partial t}, \theta \rangle &= -\langle u_m, \frac{\partial \theta}{\partial t} \rangle \\ &= -\int_{-\infty}^{\infty} \int_{\mathbb{R}^3} E_3(\phi_m^*(\zeta_m^{1/2} v)) \tilde{\Psi}_m (\partial\theta/\partial t) (y,t) dy dt \\ &= -\int_{-\infty}^{\infty} \int_{\mathbb{R}^3} E_3(\phi_m^*(\zeta_m^{1/2} v)) \frac{\partial}{\partial t} \tilde{\Psi}_m (\theta) (y,t) dy dt \\ &= \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} \frac{\partial}{\partial t} E_3(\phi_m^*(\zeta_m^{1/2} v)) \tilde{\Psi}_m (\theta) (y,t) dy dt. \end{aligned}$$

Taking absolute values, we deduce that

$$|\langle \frac{\partial u_m}{\partial t}, \theta \rangle| \leq C(\Omega) \| \frac{\partial}{\partial t} E_3(\phi_m^*(\zeta_m^{1/2} v)) \|_{L^2(H^{-1}(\mathbb{R}^3))} \|\theta\|_{L^2(H^1(\mathbb{R}^3))},$$

which shows that

$$\frac{\partial u_m}{\partial t}(x,t) \in L^2(\mathbb{R}, H^{-1}(\mathbb{R}^3)).$$

Since we already know that $u_m(x,t)$ belong to $L^2(\mathbb{R}, H^1(\mathbb{R}^3))$, we conclude that $u_m \in V$. Finally, by the linearity of the trace operator, it follows that

$$u = \sum_{m=1}^{M} u_m, \tag{2.27}$$

belongs to V and satisfies $\gamma u = v$.

3 Mapping Properties of the Heat Operator

To study the heat operator $\Lambda = \partial/\partial t - \Delta$ on functions $u(x,t)$ which are defined for all $(x,t) \in \mathbb{R}^3 \times \mathbb{R}$, we require some special spaces of functions. As we shall show below, these spaces coincide with certain locally anisotropic Sobolev spaces and are natural generalizations of the types of spaces used by [13], [20], [28] in their treatments of the Laplace operator over unbounded regions. In this section, we first address the definition of these spaces and then establish the mapping properties of the heat operator on them.

For all $u \in \mathcal{S}(\mathbb{R}^3 \times \mathbb{R})$, set

$$\|u\|^2_{\mathcal{W}^{1,0}} = \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} |\xi|^2 |\hat{u}(\xi,\tau)|^2 d\xi d\tau, \tag{3.1}$$

$$\|u\|^2_{\mathcal{W}^{1,1/2}} = \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} (|\xi|^2 + |\tau|) |\hat{u}(\xi,\tau)|^2 d\xi d\tau, \tag{3.2}$$

$$\|u\|^2_{\mathcal{V}} = \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} \left(|\xi|^2 + |\tau|^2 |\xi|^{-2}\right) |\hat{u}(\xi,\tau)|^2 d\xi d\tau, \tag{3.3}$$

where $\hat{u}$ denotes the Fourier transform of u in space and time. Since the weight functions in (3.1–3.3) are each locally integrable over $\mathbb{R}^3 \times \mathbb{R}$, it follows that each of these expression is finite for all $u \in \mathcal{S}(\mathbb{R}^3 \times \mathbb{R})$. The estimates

$$\|u\|_{\mathcal{W}^{1,0}} \leq \|u\|_{\mathcal{W}^{1,1/2}} \leq 2\|u\|_{\mathcal{V}}, \qquad u \in \mathcal{S}(\mathbb{R}^3 \times \mathbb{R}), \tag{3.4}$$

are easily derived. We denote by $\mathcal{W}^{1,0}$, $\mathcal{W}^{1,1/2}$, and $\mathcal{V}$ the completions of $\mathcal{S}(\mathbb{R}^3 \times \mathbb{R})$ in the respective norms given by (3.1-3.3). Thus we have the dense continuous inclusions

$$\mathcal{S}(\mathbb{R}^3 \times \mathbb{R}) \subset \mathcal{V} \subset \mathcal{W}^{1,1/2} \subset \mathcal{W}^{1,0}.$$

Our first task is to show that these spaces can be identified with a space of locally integrable functions. We will show this using the fact that the norms given by (3.1–3.3) differ from the norms over the anisotropic Sobolev spaces $H^{1,0}$, $H^{1,1/2}$, and V,

$$\|u\|^2_{H^{1,0}} = \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} (1 + |\xi|^2) |\hat{u}(\xi,\tau)|^2 d\xi d\tau, \tag{3.5}$$

$$\|u\|^2_{H^{1,1/2}} = \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} (1 + |\xi|^2 + |\tau|) |\hat{u}(\xi,\tau)|^2 d\xi d\tau, \tag{3.6}$$

$$\|u\|^2_{V} = \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} \left(1 + |\xi|^2 + |\tau|^2 (1 + |\xi|^2)^{-1}\right) |\hat{u}(\xi,\tau)|^2 d\xi d\tau, \tag{3.7}$$

solely because their weight functions do not contain the constant term 1.

Let $\Psi(\xi)$ denote any non-negative function in $\mathcal{D}(\mathbb{R}^3)$ which satisfies $0 \leq \Psi(\xi) \leq 1$ for all $\xi \in \mathbb{R}^3$ with

$$\Psi(\xi) = \begin{cases} 1 & |\xi| \leq 1, \\ 0 & |\xi| \geq 2. \end{cases}$$

For convenience, set

$$B_2 = \{\xi \in \mathbb{R}^3 \colon |\xi| \leq 2\}.$$

Given any $u \in \mathcal{S}(\mathbb{R}^3 \times \mathbb{R})$, let $\hat{u}(\xi, \tau)$ denote its Fourier transform. We set

$$\hat{u}_1(\xi, \tau) = (1 - \Psi(\xi))\hat{u}(\xi, \tau), \tag{3.8}$$

$$\hat{u}_2(\xi, \tau) = \Psi(\xi)\hat{u}(\xi, \tau), \tag{3.9}$$

and then define $u_1(x,t)$ and $u_2(x,t)$ to be the respective inverse Fourier transforms of $\hat{u}_1(\xi, \tau)$ and $\hat{u}_2(\xi, \tau)$. Note that both u_1 and u_2 belong to $\mathcal{S}(\mathbb{R}^3 \times \mathbb{R})$ since $u \in \mathcal{S}(\mathbb{R}^3 \times \mathbb{R})$. Moreover,

$$u(x,t) = u_1(x,t) + u_2(x,t), \qquad (x,t) \in \mathbb{R}^3 \times \mathbb{R}.$$

In the next lemma, we establish useful estimates satisfied by the functions $u_1(x,t)$ and $u_2(x,t)$. In stating this lemma, we denote by $W^{1,\infty}(\mathbb{R}^3)$ the Banach space

$$W^{1,\infty}(\mathbb{R}^3) = \{\theta(x) \in L^\infty \colon |\nabla\theta(x)| \in L^\infty(\mathbb{R}^3)\},$$

with associated norm

$$\|\theta\|_{W^{1,\infty}(\mathbb{R}^3)} = \sup_{x \in \mathbb{R}^3} \left(|\theta(x)| + |\nabla\theta(x)|\right).$$

Lemma 3.1 *For any $u \in \mathcal{S}(\mathbb{R}^3 \times \mathbb{R})$, let $u_1(x,t)$ and $u_2(x,t)$ be defined as above. Then,*

$$\begin{aligned} \|u_1\|_{H^{1,0}} &\leq 2\|u\|_{\mathcal{W}^{1,0}}, \\ \|u_1\|_{H^{1,1/2}} &\leq 2\|u\|_{\mathcal{W}^{1,1/2}}, \\ \|u_1\|_{V} &\leq 2\|u\|_{\mathcal{V}}, \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} \|u_2\|_{L^2(\mathbb{R}, W^{1,\infty}(\mathbb{R}^3))} &\leq C\|u\|_{\mathcal{W}^{1,0}}, \\ \|u_2\|_{H^{1/2}(\mathbb{R}, L^\infty(\mathbb{R}^3))} &\leq C\|u\|_{\mathcal{W}^{1,1/2}}, \\ \|u_2\|_{H^{1}(\mathbb{R}, L^\infty(\mathbb{R}^3))} &\leq C\|u\|_{\mathcal{V}}. \end{aligned} \tag{3.11}$$

Proof: Since $\hat{u}_1(\xi,\tau)$ is zero in a neighborhood of $|\xi|=0$, it is a simple matter to show (3.10). To show (3.11), we use the fact that the inverse Fourier transform $\mathcal{F}_x^{-1}$ maps $L^1(\mathbb{R}^3)$ boundedly into $L^\infty(\mathbb{R}^3)$. Thus, to show the first estimate of (3.11), it suffices to show

$$\int_{-\infty}^{\infty} \|(1+|\xi|)\hat{u}_2(\cdot,\tau)\|^2_{L^1(\mathbb{R}^3)} d\tau \le C\|u\|^2_{\mathcal{W}^{1,0}}. \tag{3.12}$$

Since $\Psi(\xi)$ has compact support in B_2 and is bounded in absolute value by unity, we have

$$\begin{aligned} \|(1+|\xi|)\hat{u}_2(\xi,\tau)\|_{L^1(\mathbb{R}^3)} &= \|\,(1+|\xi|^{-1})|\xi|\hat{u}_2(\xi,\tau)\,\|_{L^1(B_2)} \\ &\le \|\,(1+|\xi|^{-1})|\xi|\hat{u}(\xi,\tau)\,\|_{L^1(B_2)}. \end{aligned} \tag{3.13}$$

Applying the Cauchy-Schwartz inequality to the right of (3.13), we get

$$\|(1+|\xi|)\hat{u}_2(\xi,\tau)\|_{L^1(\mathbb{R}^3)} \le \|\,1+|\xi|^{-1}\,\|_{L^2(B_2)}\|\,|\xi|\hat{u}(\xi,\tau)\,\|_{L^2(\mathbb{R}^3)}.$$

Since $|\xi|^{-2}$ is locally integrable in $\mathbb{R}^3$, it follows that

$$\|(1+|\xi|)\hat{u}_2(\xi,\tau)\|_{L^1(\mathbb{R}^3)} \le C\|\,|\xi|\hat{u}(\xi,\tau)\,\|_{L^2(\mathbb{R}^3)}.$$

Squaring both sides of this inequality and integrating over τ, we obtain (3.12).

To show the second estimate in (3.11), it suffices to combine (3.12) with the inequality

$$\int_{-\infty}^{\infty} |\tau|\,\|\hat{u}_2(\cdot,\tau)\|^2_{L^1(\mathbb{R}^3)} d\tau \le C\|u\|^2_{\mathcal{W}^{1,1/2}},$$

which follows from the trivial bound

$$\|\hat{u}_2(\cdot,\tau)\|_{L^1(\mathbb{R}^3)} \le C\|\hat{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)} \quad \text{for all} \quad \tau\in\mathbb{R}.$$

Since

$$\mathcal{F}_{x,t}\left(\frac{\partial u_2}{\partial t}\right)(\xi,\tau) = i\tau\hat{u}_2(\xi,\tau), \qquad (\xi,\tau)\in\mathbb{R}^3\times\mathbb{R},$$

the final estimate of (3.11) is verified by combining (3.12) with

$$\begin{aligned} \int_{-\infty}^{\infty} |\tau|^2\,\|\hat{u}_2(\cdot,\tau)\|^2_{L^1(\mathbb{R}^3)} d\tau &\le C\int_{-\infty}^{\infty} |\tau|^2\,\|\,|\xi|^{-1}\hat{u}_2(\cdot,\tau)\,\|^2_{L^2(\mathbb{R}^3)} d\tau \\ &\le C\|u\|^2_{\mathcal{V}}. \qquad \square \end{aligned}$$

Based on lemma 3.1, we can now show that each of the completed spaces $\mathcal{W}^{1,0}$, $\mathcal{W}^{1,1/2}$, and $\mathcal{V}$ may be identified with a space of locally integrable functions. Specifically, in the case of $\mathcal{W}^{1,0}$, consider the Banach space

$$X = H^{1,0} + L^2(\mathbb{R}, W^{1,\infty}(\mathbb{R}^3)),$$

with associated norm

$$\|u\|_X^2 = \inf_{u=u_1+u_2} \left(\|u_1\|_{H^{1,0}}^2 + \|u_2\|_{L^2(\mathbb{R},W^{1,\infty}(\mathbb{R}^3))}^2 \right).$$

The estimates given in lemma 3.1 immediately show that any sequence $\{u_k\}$ of functions $u_k \in \mathcal{S}(\mathbb{R}^3 \times \mathbb{R})$ which is a Cauchy sequence in the $\mathcal{W}^{1,0}$ norm is also a Cauchy sequence in X. Since X is complete, there exists a unique $u \in X$ which is the limit of the sequence $\{u_k\}$. It is readily checked that equivalent Cauchy sequences, i.e., two sequences $\{u_k\}$ and $\{u'_k\}$ such that

$$\lim_{k\to\infty} \|u_k - u'_k\|_{\mathcal{W}^{1,0}} = 0,$$

define the same $u \in X$. Thus, we can identify $\mathcal{W}^{1,0}$ with a subspace of X. By similar considerations applied to $\mathcal{W}^{1,1/2}$ and $\mathcal{V}$, we can make these identifications:

$$\begin{aligned}
\mathcal{W}^{1,0} &\subset H^{1,0} + L^2(\mathbb{R}, W^{1,\infty}(\mathbb{R}^3)), \\
\mathcal{W}^{1,1/2} &\subset H^{1,1/2} + L^2(\mathbb{R}, W^{1,\infty}(\mathbb{R}^3)) \cap H^{1/2}(\mathbb{R}, L^\infty(\mathbb{R}^3)), \\
\mathcal{V} &\subset V + L^2(\mathbb{R}, W^{1,\infty}(\mathbb{R}^3)) \cap H^1(\mathbb{R}, L^\infty(\mathbb{R}^3)).
\end{aligned}$$

We remark that these inclusions are strict. They, however, clearly imply the next theorem which shows that the spaces $\mathcal{W}^{1,0}$, $\mathcal{W}^{1,1/2}$, and $\mathcal{V}$ differ from the anisotropic Sobolev spaces $H^{1,0}$, $H^{1,1/2}$, and V solely in their permitted behavior as x tends to infinity.

Theorem 3.2 *For any $\theta \in \mathcal{S}(\mathbb{R}^3)$, there exists a constant $C(\theta)$ such that*

$$\|\theta u\|_{H^{1,0}} \leq C(\theta)\|u\|_{\mathcal{W}^{1,0}}, \tag{3.14}$$

$$\|\theta u\|_{H^{1,1/2}} \leq C(\theta)\|u\|_{\mathcal{W}^{1,1/2}}, \tag{3.15}$$

$$\|\theta u\|_{V} \leq C(\theta)\|u\|_{\mathcal{V}}, \tag{3.16}$$

for all $u \in \mathcal{S}(\mathbb{R}^3 \times \mathbb{R})$.

The importance of Theorem 3.2 is that we can immediately extend the trace operator γ to the spaces $\mathcal{W}^{1,1/2}$ and $\mathcal{V}$. Furthermore, we can conclude the following crucial theorem.

Theorem 3.3 *Let γ: $\mathcal{W}^{1,1/2} \to H^{1/2,1/4}(\Gamma,\mathbb{R})$ denote the trace operator. Then, $\gamma_+ = R_+\gamma$ maps $\mathcal{V}$ onto $H^{1/2,1/4}(\Gamma,\mathbb{R})$.*

Now, the time restriction operator R_+: $H^{1/2,1/4}(\Gamma,\mathbb{R}) \to H^{1/2,1/4}(\Gamma,\mathbb{R}_+)$ is onto. Combining this fact with the above theorem gives the following corollary.

Corollary 3.4 *The trace operator $\gamma_+ = R_+\gamma$ maps $\mathcal{V}$ onto $H^{1/2,1/4}(\Gamma,\mathbb{R}_+)$.*

Denote by $\mathcal{W}^{-1,0}$, $\mathcal{W}^{-1,-1/2}$, and $\mathcal{V}^*$ the respective dual spaces of $\mathcal{W}^{1,0}$, $\mathcal{W}^{1,1/2}$, and $\mathcal{V}$. These are each spaces of tempered distributions which satisfy the inclusions

$$\mathcal{W}^{-1,0} \subset \mathcal{W}^{-1,-1/2} \subset \mathcal{V}^* \subset \mathcal{S}'(\mathbb{R}^3 \times \mathbb{R}).$$

Norms over these spaces may be defined using Fourier transforms as

$$\begin{aligned}
\|f\|^2_{\mathcal{W}^{-1,0}} &= \int_{-\infty}^{\infty}\int_{\mathbb{R}^3} \frac{|\hat{f}(\xi,\tau)|^2}{|\xi|^2} d\xi d\tau, \\
\|f\|^2_{\mathcal{W}^{-1,-1/2}} &= \int_{-\infty}^{\infty}\int_{\mathbb{R}^3} \frac{|\hat{f}(\xi,\tau)|^2}{|\xi|^2+|\tau|} d\xi d\tau, \\
\|f\|^2_{\mathcal{V}^*} &= \int_{-\infty}^{\infty}\int_{\mathbb{R}^3} \frac{|\xi|^2}{|\xi|^4+|\tau|^2} |\hat{f}(\xi,\tau)|^2 d\xi d\tau.
\end{aligned}$$

The mapping properties of the heat operator Λ on $\mathcal{W}^{1,1/2}$ and $\mathcal{W}^{1,0}$ is the subject of the next theorem.

Theorem 3.5 *The heat operator Λ extends from $\mathcal{S}(\mathbb{R}^3 \times \mathbb{R})$ to an isomorphism of $\mathcal{W}^{1,1/2}$ onto $\mathcal{W}^{-1,-1/2}$ and of $\mathcal{W}^{1,0}$ onto $\mathcal{V}^*$. Moreover,*

$$Re\,\langle \Lambda u, u\rangle = \|u\|^2_{\mathcal{W}^{1,0}} \quad \text{for all} \quad u \in \mathcal{W}^{1,1/2}. \tag{3.17}$$

Proof: The theorem is an easy consequence of the equality

$$\mathcal{F}_{x,t}(\Lambda u) = (i\tau + |\xi|^2)\hat{u}(\xi,\tau), \qquad u \in \mathcal{S}(\mathbb{R}^3 \times \mathbb{R}),$$

and the density of $\mathcal{S}(\mathbb{R}^3 \times \mathbb{R})$ in these spaces. □

We now give an an alternative proof of the isomorphism of Λ: $\mathcal{W}^{1,1/2} \to \mathcal{W}^{-1,-1/2}$. We do so because this method of proof extends to Dirichlet problems, which we address in section 4. For all $u, v \in \mathcal{S}(\mathbb{R}^3 \times \mathbb{R})$, let

$$\mathcal{B}(u,v) = \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} \frac{\partial u}{\partial t}(x,t)\overline{v(x,t)} + \nabla u(x,t)\overline{\nabla v(x,t)} dx dt, \tag{3.18}$$

where the bar denotes complex conjugation. By Parseval's theorem,

$$\mathcal{B}(u,v) = \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} i\tau \hat{u}(\xi,\tau)\overline{\hat{v}(\xi,\tau)} + |\xi|^2 \hat{u}(\xi,\tau)\overline{\hat{v}(\xi,\tau)} d\xi d\tau \tag{3.19}$$

A simple application of Cauchy-Schwartz's inequality shows that $\mathcal{B}$ satisfies

$$|\mathcal{B}(u,v)| \le \|u\|_{\mathcal{W}^{1,1/2}} \|v\|_{\mathcal{W}^{1,1/2}}$$

for all $u, v \in \mathcal{S}(\mathbb{R}^3 \times \mathbb{R})$. Therefore, $\mathcal{B}$ uniquely extends to a bounded sesquilinear form on $\mathcal{W}^{1,1/2} \times \mathcal{W}^{1,1/2}$. In the next lemma, we show that this extension satisfies conditions which readily imply that Λ is an isomorphism.

Lemma 3.6 *There exists a bounded, linear operator $\mathcal{H}$: $\mathcal{W}^{1,1/2} \to \mathcal{W}^{1,1/2}$ such that*

$$Re\ \mathcal{B}(u, u - \mathcal{H}u) = \|u\|^2_{\mathcal{W}^{1,1/2}} \quad \text{for all} \quad u \in \mathcal{W}^{1,1/2}, \tag{3.20}$$

and

$$Re\ \mathcal{B}(v + \mathcal{H}v, v) = \|v\|^2_{\mathcal{W}^{1,1/2}} \quad \text{for all} \quad v \in \mathcal{W}^{1,1/2}. \tag{3.21}$$

Proof: For each $u \in \mathcal{W}^{1,1/2}$, let $\mathcal{H}u(x,t)$ denote the function defined by

$$\mathcal{F}_{x,t}(\mathcal{H}u)(\xi,\tau) = -i\text{sign}(\tau)\hat{u}(\xi,\tau), \qquad (\xi,\tau) \in \mathbb{R}^3 \times \mathbb{R}. \tag{3.22}$$

This defines a mapping $\mathcal{H}$ which is an isometry on $\mathcal{W}^{1,1/2}$. This map is essentially the Hilbert transform in time. Readily, we see that

$$\mathcal{B}(u,u) = \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} i\tau |\hat{u}(\xi,\tau)|^2 + |\xi|^2 |\hat{u}(\xi,\tau)|^2 d\xi d\tau. \tag{3.23}$$

and

$$\mathcal{B}(u,\mathcal{H}u) = \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} -|\tau|\, |\hat{u}(\xi,\tau)|^2 + i\text{sign}(\tau)|\xi|^2 |\hat{u}(\xi,\tau)|^2 d\xi d\tau. \tag{3.24}$$

By subtracting (3.24) from (3.23) and then taking real parts, we get

$$\mathrm{Re}\ \mathcal{B}(u, u - \mathcal{H}u) = \|u\|^2_{\mathcal{W}^{1,1/2}}.$$

Equation (3.21) is verified analogously. □

This lemma implies [4, Theorem 5.1.2] that $\Lambda\colon \mathcal{W}^{1,1/2} \to \mathcal{W}^{-1,-1/2}$ defined by

$$\langle \Lambda u, v \rangle = \mathcal{B}(u, v) \quad \text{for all} \quad v \in \mathcal{W}^{1,1/2}, \tag{3.25}$$

is an isomorphism.

4 Mapping Properties of the Single Layer Heat Potential

In this section, we set out the mapping properties of the single layer heat potential. We recall our notation for this operator

$$\mathcal{K}_1 q(x,t) = \int_0^t \int_\Gamma K(x-x', t-t') q(x',t') dx' dt', \quad (x,t) \in \Gamma \times \mathbb{R}_+. \qquad (4.1)$$

Let $C_+^\infty(\Gamma \times \mathbb{R}_+)$ denote the space of functions obtained by restricting functions $\tilde{q}(x,t) \in \mathcal{D}(\Gamma \times \mathbb{R})$ to $t \geq 0$. In Theorems 4.2 and 4.3 below, we will show that $\mathcal{K}_1$ extends from an operator on $C_+^\infty(\Gamma \times \mathbb{R}_+)$ to an isomorphism from $H^{-1/2,-1/4}(\Gamma, \mathbb{R}_+)$ onto $H^{1/2,1/4}(\Gamma, \mathbb{R}_+)$. First, we need a preliminary lemma.

Lemma 4.1 *For all $q \in C_+^\infty(\Gamma \times \mathbb{R}_+)$, let δ_q denote the tempered distribution defined by*

$$\langle \delta_q, \theta \rangle = \int_0^\infty \int_\Gamma q(x',t') \theta(x',t') dx' dt' \quad \textit{for all} \quad \theta \in \mathcal{S}(\mathbb{R}^3 \times \mathbb{R}).$$

Then, with K denoting the fundamental solution to the heat equation and $\star$ convolution in the sense of distributions, we have $K \star \delta_q \in \mathcal{S}'(\mathbb{R}^3 \times \mathbb{R})$. Furthermore,

$$\mathcal{F}_{x,t}\left(K \star \delta_q\right)(\xi, \tau) = \frac{\mathcal{F}_{x,t}\left(\delta_q\right)(\xi,\tau)}{\xi^2 + i\tau}, \qquad (4.2)$$

in the sense of tempered distributions.

Proof: Since q equals zero for large t, δ_q is a distribution with compact support. Because $K \in \mathcal{S}'(\mathbb{R}^3 \times \mathbb{R})$, standard results of distribution theory (cf., [40, pp. 178-179],) show that $K \star \delta_q \in \mathcal{S}'(\mathbb{R}^3 \times \mathbb{R})$ and that (4.2) holds. □

We now establish a basic factorization result which equates the operator $\mathcal{K}_1$ on $C_+^\infty(\Gamma, \mathbb{R}_+)$ with an operator whose extension to $H^{-1/2,-1/4}(\Gamma, \mathbb{R}_+)$ is immediate.

Theorem 4.2 *Let Λ: $\mathcal{W}^{1,1/2} \to \mathcal{W}^{-1,-1/2}$ denote the heat operator and γ_+: $\mathcal{W}^{1,1/2} \to H^{1/2,1/4}(\Gamma, \mathbb{R}_+)$ the trace operator. The integral operator $\mathcal{K}_1$ defined on $C_+^\infty(\Gamma \times \mathbb{R}_+)$ by (4.1) agrees with the composition $\gamma_+ \circ \Lambda^{-1} \circ \gamma_+^*$ and consequently extends to a bounded linear operator of $H^{-1/2,-1/4}(\Gamma, \mathbb{R}_+)$ into $H^{1/2,1/4}(\Gamma, \mathbb{R}_+)$.*

Proof: Since γ_+: $\mathcal{W}^{1,1/2} \to H^{1/2,1/4}(\Gamma, \mathbb{R}_+)$ is bounded, its adjoint γ_+^* maps $H^{-1/2,-1/4}(\Gamma, \mathbb{R}_+)$ boundedly into $\mathcal{W}^{-1,-1/2}$. For each $q \in H^{-1/2,-1/4}(\Gamma, \mathbb{R}_+)$, define $u_q = \Lambda^{-1}\gamma_+^* q$. Since Λ: $\mathcal{W}^{1,1/2} \to \mathcal{W}^{-1,-1/2}$ is an isomorphism, we have

$$\|u_q\|_{\mathcal{W}^{1,1/2}} \leq C\|q\|_{H^{-1/2,-1/4}(\Gamma,\mathbb{R}_+)}. \tag{4.3}$$

By Theorem 3.5, the Fourier transform of u_q is

$$\mathcal{F}_{x,t}\left(u_q\right)(\xi,\tau) = \frac{\mathcal{F}_{x,t}\left(\gamma_+^* q\right)(\xi,\tau)}{|\xi|^2 + i\tau}, \qquad (\xi,\tau) \in \mathbb{R}^3 \times \mathbb{R}. \tag{4.4}$$

Now, suppose that $q \in C_+^\infty(\Gamma \times \mathbb{R}_+)$. Since $\gamma_+^* q$ is defined by

$$\langle \gamma_+^* q, v\rangle = \langle q, \gamma_+ v\rangle \quad \text{for all} \quad v \in \mathcal{W}^{1,1/2}, \tag{4.5}$$

it follows that $\gamma_+^* q$ agrees with δ_q (defined in Lemma 4.1). Therefore, by Lemma 4.1, we have

$$\begin{aligned} u_q(x,t) &= K \star \delta_q \\ &= \int_0^t \int_\Gamma K(x-x', t-t')q(x',t')dx'dt', \qquad (x,t) \in \mathbb{R}^3 \times \mathbb{R}. \end{aligned}$$

Taking the trace of u_q, we get

$$\gamma_+ \circ \Lambda^{-1} \circ \gamma_+^* q = \int_0^t \int_\Gamma K(x-x', t-t')q(x',t')dx'dt', \quad (x,t) \in \Gamma \times \mathbb{R}_+. \tag{4.6}$$

Thus, (4.6) shows that on $C_+^\infty(\Gamma \times \mathbb{R}_+)$ the single layer heat potential $\mathcal{K}_1$ coincides with the bounded operator $\gamma_+\Lambda^{-1}\gamma_+^*$. Because $C_+^\infty(\Gamma \times \mathbb{R}_+)$ is dense in $H^{-1/2,-1/4}(\Gamma \times \mathbb{R}_+)$, the operator $\mathcal{K}_1$ uniquely extends to $H^{-1/2,-1/4}(\Gamma \times \mathbb{R}_+)$. □

We now give one of the main results of this paper.

Theorem 4.3 *The single layer heat potential* $\mathcal{K}_1 = \gamma_+ \circ \Lambda^{-1} \circ \gamma_+^*$ *is an isomorphism of* $H^{-1/2,-1/4}(\Gamma, \mathbb{R}_+)$ *onto* $H^{1/2,1/4}(\Gamma, \mathbb{R}_+)$*. Furthermore, it satisfies the coercivity estimate*

$$Re\ \langle q, \mathcal{K}_1 q\rangle \geq c\|q\|^2_{H^{-1/2,-1/4}(\Gamma,\mathbb{R}_+)}, \tag{4.7}$$

for some positive constant c.

Proof: We consider the duality pairing $\langle q, \mathcal{K}_1 q\rangle$ for $q \in H^{-1/2,-1/4}(\Gamma, \mathbb{R}_+)$. Using the definition of γ_+^*, we have

$$\langle q, \mathcal{K}_1 q\rangle = \langle \gamma_+^* q, \Lambda^{-1}\gamma_+^* q\rangle.$$

Now, since $\gamma_+ \colon \mathcal{V} \to H^{1/2,1/4}(\Gamma, \mathbb{R}_+)$ is surjective, it follows (see [40, Theorems 4.12 and 4.14]) that its adjoint γ_+^* maps $H^{-1/2,-1/4}(\Gamma, \mathbb{R}_+)$ isomorphically onto a closed subspace of $\mathcal{V}^*$. Hence, there exists some positive constant c such that

$$\|\gamma_+^* q\|_{\mathcal{V}^*} \geq c\|q\|_{H^{-1/2,-1/4}(\Gamma,\mathbb{R}_+)} \quad \text{for all} \quad q \in H^{-1/2,-1/4}(\Gamma, \mathbb{R}_+). \tag{4.8}$$

The argument

$$\begin{aligned}
\text{Re}\, \langle q, \mathcal{K}_1 q\rangle = &\ \text{Re}\, \langle \Lambda\Lambda^{-1}\gamma_+^* q, \Lambda^{-1}\gamma_+^* q\rangle \\
\geq &\ \|\Lambda^{-1}\gamma_+^* q\|^2_{\mathcal{W}^{1,0}} && \text{by (3.17)}, \\
\geq &\ c\|\gamma_+^* q\|^2_{\mathcal{V}^*} && \text{by Theorem 3.5}, \\
\geq &\ c\|q\|^2_{H^{-1/2,-1/4}(\Gamma,\mathbb{R}_+)}, && \text{by (4.8)},
\end{aligned}$$

then proves the coercivity statement. By the Lax-Milgram theorem, $\mathcal{K}_1$ is an isomorphism. □

In studying the regularity of the single layer operator $\mathcal{K}_1$, it will be convenient to work mainly with the operator $\tilde{\mathcal{K}}_1 = \gamma\Lambda^{-1}\gamma^*$. We have

$$\mathcal{K}_1 = R_+ \tilde{\mathcal{K}}_1 R_+^*,$$

where $R_+ \colon H^{1/2,1/4}(\Gamma, \mathbb{R}) \to H^{1/2,1/4}(\Gamma, \mathbb{R}_+)$ denotes the time restriction operator. Clearly, $\tilde{\mathcal{K}}_1$ defines an isomorphism of $H^{-1/2,-1/4}(\Gamma, \mathbb{R})$ onto $H^{1/2,1/4}(\Gamma, \mathbb{R})$. For later use, we note that this implies that the adjoint operator $\tilde{\mathcal{K}}_1^* = \gamma(\Lambda^*)^{-1}\gamma^*$ is also an isomorphism of $H^{-1/2,-1/4}(\Gamma, \mathbb{R})$ onto $H^{1/2,1/4}(\Gamma, \mathbb{R})$. The map $\tilde{\mathcal{K}}_1^*$ extends the backward heat potential

$$\tilde{\mathcal{K}}_1^* p(x,t) = \int_t^\infty \int_\Gamma K(x - x', t' - t) p(x', t') dx' dt', \qquad (x,t) \in \mathbb{R}^3 \times \mathbb{R}.$$

We now adapt an argument due to Nedelec and Planchard which further characterizes the inverse operator $\tilde{\mathcal{K}}_1^{-1}$. The starting point is a variational formulation of the homogeneous Dirichlet problem

$$\frac{\partial u}{\partial t} - \Delta u = f \quad \text{on} \quad \mathbb{R}^3 \backslash \Gamma \times \mathbb{R} \tag{4.9}$$

$$u = 0 \quad \text{on} \quad \Gamma \times \mathbb{R}. \tag{4.10}$$

To treat this problem, let

$$\mathcal{W}_\Gamma^{1,1/2} = \{u \in \mathcal{W}^{1,1/2}: \gamma u = 0\}.$$

Thus, $\mathcal{W}_\Gamma^{1,1/2}$ corresponds to the subspace of $\mathcal{W}^{1,1/2}$ functions $u(x,t)$ which vanish for $x \in \Gamma$. Equivalently, $\mathcal{W}_\Gamma^{1,1/2}$ is the closure in the $\mathcal{W}^{1,1/2}$ norm of $\mathcal{S}(\mathbb{R}^3 \backslash \Gamma \times \mathbb{R})$. (Note therefore that the dual space $(W_\Gamma^{1,1/2})^*$ is contained in $\mathcal{D}(\mathbb{R}^3 \backslash \Gamma \times \mathbb{R})$ but *not* in $\mathcal{D}(\mathbb{R}^3 \times \mathbb{R})$.) Since $\mathcal{W}_\Gamma^{1,1/2}$ is a subspace of $\mathcal{W}^{1,1/2}$, the bilinear form $\mathcal{B}$ introduced in (3.19)

$$\mathcal{B}(u,v) = \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} (i\tau + |\xi|^2) \hat{u}(\xi,\tau) \overline{\hat{v}(\xi,\tau)} d\xi d\tau,$$

is well defined for all $u, v \in \mathcal{W}_\Gamma^{1,1/2}$. Thus, for any $f \in (\mathcal{W}_\Gamma^{1,1/2})^*$, we may consider the variational problem of finding $u \in \mathcal{W}_\Gamma^{1,1/2}$ such that

$$\mathcal{B}(u,v) = \langle f, v \rangle \quad \text{for all} \quad v \in \mathcal{W}_\Gamma^{1,1/2} \tag{4.11}$$

A standard argument shows how Problem (4.11) extends the Dirichlet problem (4.9–4.10). Let u denote any smooth solution u to problem (4.11). By a smooth solution to (4.11), we mean a $C(\mathbb{R}^3 \times \mathbb{R})$ function u (vanishing sufficiently rapidly at infinity say) which is twice continuously differentiable in each of the sets $\Omega \times \mathbb{R}$ and $\Omega^c \times \mathbb{R}$. Clearly, for such a function, we have

$$\begin{aligned} \mathcal{B}(u,v) &= \int_{-\infty}^{\infty} \int_{\Omega} \left(\frac{\partial u}{\partial t}(x,t) - \Delta u(x,t) \right) v(x,t)\, dx\, dt \\ &+ \int_{-\infty}^{\infty} \int_{\Omega^c} \left(\frac{\partial u}{\partial t}(x,t) - \Delta u(x,t) \right) v(x,t)\, dx\, dt, \end{aligned}$$

for any v in the Schwartz class $\mathcal{S}(\mathbb{R}^3 \times \mathbb{R})$. Hence, by Green's theorem,

$$\begin{aligned}\mathcal{B}(u,v) &= \int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\left(\frac{\partial u}{\partial t}(x,t) - \Delta u(x,t)\right)v(x,t)\,dx\,dt \\ &\quad + \int_{-\infty}^{\infty}\int_{\Gamma}\left(\frac{\partial u}{\partial \mathbf{n}}(x,t)\mid_{\text{int}} - \frac{\partial u}{\partial \mathbf{n}}(x,t)\mid_{\text{ext}}\right)v(x,t)dxdt, \\ &\qquad v \in \mathcal{S}(\mathbb{R}^3 \times \mathbb{R}).\end{aligned}$$

Since u by assumption solves (4.11), it follows that

$$\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\left(\frac{\partial u}{\partial t}(x,t) - \Delta u(x,t)\right)v(x,t)\,dx\,dt = \int_{-\infty}^{\infty}\int_{\mathbb{R}^3} f(x,t)v(x,t)\,dx\,dt,$$

for any arbitrary function $v \in \mathcal{S}(\mathbb{R}^3 \times \mathbb{R})$ which vanishes on the boundary Γ. Hence, u satisfies (4.9).

Lemma 4.4 *The variational problem (4.11) defines an isomorphism of $u \in \mathcal{W}_\Gamma^{1,1/2}$ onto $f \in \left(\mathcal{W}_\Gamma^{1,1/2}\right)^*$ and consequently extends the homogeneous Dirichlet problem (4.9–4.10).*

Proof: As in the proof of Lemma 3.6, let $\mathcal{H}$ denote the Hilbert transform defined by

$$\mathcal{F}_{x,t}(\mathcal{H}u)(\xi,\tau) = -i\text{sign}(\tau)\mathcal{F}_{x,t}(u)(\xi,\tau) \quad \text{for all} \quad u \in \mathcal{S}'(\mathbb{R}^3 \times \mathbb{R}).$$

Already, we have shown that

$$\begin{aligned}&\mathcal{B}(u,(I-\mathcal{H})u) \;\geq\; \|u\|^2_{\mathcal{W}^{1,1/2}} \quad \text{for all} \quad u \in \mathcal{W}^{1,1/2}, \\ \text{and}\quad & \\ &\mathcal{B}((I+\mathcal{H})v,v) \;\geq\; \|v\|^2_{\mathcal{W}^{1,1/2}} \quad \text{for all} \quad v \in \mathcal{W}^{1,1/2}.\end{aligned} \tag{4.12}$$

Since the Hilbert transform is an operator only in time, it is easy to see that $\mathcal{H}$ maps $\mathcal{W}_\Gamma^{1,1/2}$ onto $\mathcal{W}_\Gamma^{1,1/2}$. This along with (4.12) clearly implies the lemma. $\square$

Using trace theory, we extend this result to a Dirichlet problem with inhomogeneous boundary data in the next theorem. Without loss of generality, we assume that the forcing function f equals zero. (We can always add to solutions given below a function u_f determined by lemma 4.4 which vanishes on the boundary and satisfies (4.9).)

Theorem 4.5 *For each $g \in H^{1/2,1/4}(\Gamma, \mathbb{R})$, there exists a unique solution $u \in \mathcal{W}^{1,1/2}$ to the Dirichlet problem*

$$\frac{\partial u}{\partial t} - \Delta u = 0 \quad on \quad \mathbb{R}^3 \backslash \Gamma \times \mathbb{R}, \tag{4.13}$$

$$u = g \quad on \quad \Gamma \times \mathbb{R}. \tag{4.14}$$

Proof: Given $g \in H^{1/2,1/4}(\Gamma, \mathbb{R})$, we may find $u_g \in \mathcal{W}^{1,1/2}$ such that $\gamma u_g = g$. If we let $u_0 \in \mathcal{W}_\Gamma^{1,1/2}$ denote the unique solution to

$$\mathcal{B}(u_0, v) = -\mathcal{B}(u_g, v) \quad \text{for all} \quad v \in \mathcal{W}_\Gamma^{1,1/2},$$

we then obtain a solution $u = u_g + u_0 \in \mathcal{W}^{1,1/2}$ with $\gamma u = g$. Noting that u satisfies

$$\mathcal{B}(u, v) = 0 \quad \text{for all} \quad v \in \mathcal{W}_\Gamma^{1,1/2}, \tag{4.15}$$

it easily follows that this solution is unique, that is, independent of the extension of g to $\mathcal{W}^{1,1/2}$. □

We continue to denote by u the solution to the Dirichlet problem (4.13–4.14). We will define an interior and exterior normal derivative of u in the function space $H^{-1/2,-1/4}(\Gamma, \mathbb{R})$. First, we define the spaces $\mathcal{W}^{1,1/2}(\Omega^c \times \mathbb{R})$ and $\mathcal{W}^{1,1/2}(\Omega \times \mathbb{R})$. Let $R(\Omega^c)$ denote the restriction operator $R(\Omega^c)$: $\mathcal{S}(\mathbb{R}^3 \times \mathbb{R}) \to C^\infty(\Omega^c \times \mathbb{R})$. By density, $R(\Omega^c)$ extends to $\mathcal{W}^{1,1/2}$. We now define $\mathcal{W}^{1,1/2}(\Omega^c, \mathbb{R})$ to be the image of $R(\Omega^c)$ on $\mathcal{W}^{1,1/2}$. We analogously define $\mathcal{W}^{1,1/2}(\Omega, \mathbb{R})$. Poincaré's lemma shows that this latter space coincides with $H^{1,1/2}(\Omega, \mathbb{R})$. We set

$$\begin{aligned} \mathcal{B}(u, v; \Omega^c) \;\; = \;\; & \int_{-\infty}^{\infty} \int_{\Omega^c} i\tau \mathcal{F}_t(u)(x,\tau)\overline{\mathcal{F}_t(v)(x,\tau)} \\ & \qquad + \nabla \mathcal{F}_t(u)(x,\tau)\overline{\nabla \mathcal{F}_t(v)(x,\tau)} dx d\tau, \\ & \text{for all} \quad u, v \in \mathcal{W}^{1,1/2}(\Omega^c, \mathbb{R}), \end{aligned}$$

and define $\mathcal{B}(u, v; \Omega)$ analogously.

Now, let u represent the solution to (4.13–4.14) and denote by u_+ and u_- its restrictions to Ω and Ω^c. We define linear functionals $\partial u_+/\partial \mathbf{n}$ and $\partial u_-/\partial \mathbf{n}$ over $H^{1/2,1/4}(\Gamma, \mathbb{R})$ as

$$\langle \frac{\partial u_+}{\partial \mathbf{n}}, g \rangle = \mathcal{B}(u, \mathcal{E}g; \Omega) \quad \text{for all} \quad g \in H^{1/2,1/4}(\Gamma, \mathbb{R}), \tag{4.16}$$

$$\langle \frac{\partial u_-}{\partial \mathbf{n}}, g \rangle = \mathcal{B}(u, \mathcal{E}g; \Omega^c) \quad \text{for all} \quad g \in H^{1/2,1/4}(\Gamma, \mathbb{R}), \tag{4.17}$$

where $\mathcal{E}g$ denotes any bounded extension of $g \in H^{1/2,1/4}(\Gamma, \mathbb{R})$ to $\mathcal{W}^{1,1/2}$. Because u solves (4.13–4.14) and thus (4.15), it is easy to check that (4.16) and (4.17) uniquely define $\partial u_+/\partial \mathbf{n}$ and $\partial u_-/\partial \mathbf{n}$ in $H^{-1/2,-1/4}(\Gamma, \mathbb{R})$. (That is, they are independent of the choice of extension $\mathcal{E}$.) By applying Green's theorem, it can be shown that (4.16) and (4.17) agree with the classical definitions of these normal derivatives when u is a smooth solution.

Consider

$$q := \frac{\partial u_+}{\partial \mathbf{n}} - \frac{\partial u_-}{\partial \mathbf{n}}. \tag{4.18}$$

Clearly, q belongs to $H^{-1/2,-1/4}(\Gamma, \mathbb{R})$ and solves

$$\langle q, g \rangle = \mathcal{B}(u, \mathcal{E}g) \quad \text{for all} \quad g \in H^{1/2,1/4}(\Gamma, \mathbb{R}). \tag{4.19}$$

Since (4.19) holds independent of the choice of extension,

$$\langle q, \gamma v \rangle = \mathcal{B}(u, v) \quad \text{for all} \quad v \in \mathcal{W}^{1,1/2}. \tag{4.20}$$

Equivalently, $\gamma^* q = \Lambda u$, so $\gamma \Lambda^{-1} \gamma^* q = \gamma u = g$, or

$$\tilde{\mathcal{K}}_1 q = g. \tag{4.21}$$

5 Regularity of the Single Layer Operator

The objective of this section is to establish a regularity theory for the single layer operator $\mathcal{K}_1$. Most of our efforts in this section will concentrate on studying the regularity of the related operator $\tilde{\mathcal{K}}_1$ and proving Theorem 5.1. Once we prove this theorem, we shall apply the mappings properties of the restriction and extension operator to conclude regularity results for the operator $\mathcal{K}_1$.

Theorem 5.1 *For all $r \geq 0$, the operator $\tilde{\mathcal{K}}_1$ maps $H^{r-1/2,r/2-1/4}(\Gamma, \mathbb{R})$ onto $H^{r+1/2,r/2+1/4}(\Gamma, \mathbb{R})$.*

Thanks to interpolation theory [25, p. 9], we may confine our attention to non-negative integer values of r. We will denote such values by m. The proof of Theorem 5.1 naturally divides into two separate investigations. In section 5.1, we will show that the map $\tilde{\mathcal{K}}_1$: $H^{r-1/2,r/2-1/4}(\Gamma, \mathbb{R}) \to H^{r+1/2,r/2+1/4}(\Gamma, \mathbb{R})$ is bounded. In section 5.2, we will then show that this mapping is also surjective and hence an isomorphism.

5.1 Boundedness

For any non-negative integer m, let $q \in H^{m-1/2,m/2-1/4}(\Gamma, \mathbb{R})$ be given. Our goal in this subsection is to show the estimate

$$\|\tilde{\mathcal{K}}_1 q\|_{H^{m+1/2,m/2+1/4}(\Gamma,\mathbb{R})} \leq C(m)\|q\|_{H^{m-1/2,m/2-1/4}(\Gamma,\mathbb{R})},$$

for some positive constant C which depends on m. We will prove this by establishing the pair of estimates

$$\|\tilde{\mathcal{K}}_1 q\|_{H^{m/2+1/4}(\mathbb{R},L^2(\Gamma))} \leq C_1(m)\|q\|_{H^{m-1/2,m/2-1/4}(\Gamma,\mathbb{R})}, \tag{5.1}$$

and

$$\|\tilde{\mathcal{K}}_1 q\|_{L^2(\mathbb{R},H^{m+1/2}(\Gamma))} \leq C_2(m)\|q\|_{H^{m-1/2,m/2-1/4}(\Gamma,\mathbb{R})}. \tag{5.2}$$

Our first lemma addresses estimate (5.1) and considers the regularity of the operator $\tilde{\mathcal{K}}_1$ with respect to time. For convenience, we set

$$Q_s = H^{s+1/4}(\mathbb{R}, L^2(\Gamma)) \cap H^s(\mathbb{R}, H^{1/2}(\Gamma)), \qquad s \in \mathbb{R},$$

and note that

$$(Q_{-s})^* = H^{s-1/4}(\mathbb{R}, L^2(\Gamma)) + H^s(\mathbb{R}, H^{-1/2}(\Gamma)), \qquad s \in \mathbb{R}.$$

Theorem 5.2 *The single layer operator $\tilde{\mathcal{K}}_1$ maps $(Q_{-s})^*$ into Q_s for all $s \geq 0$.*

Observe that Theorem 5.2 reduces to Theorem 4.2 for $s = 0$ because $Q_0 = H^{1/2,1/4}(\Gamma, \mathbb{R})$ and

$$\begin{aligned}(Q_0)^* &= H^{-1/4}(\mathbb{R}, L^2(\Gamma)) + L^2(\mathbb{R}, H^{-1/2}(\Gamma)) \\ &= H^{-1/2,-1/4}(\Gamma, \mathbb{R}).\end{aligned}$$

This theorem implies (5.1) since

$$\|q\|_{H^{s+1/4}(\mathbb{R},L^2(\Gamma))} \leq \|q\|_{Q_s}, \qquad s \in \mathbb{R},$$

and

$$\begin{aligned}\|q\|_{(Q_{-s})^*} &\leq \|q\|_{H^{s-1/4}(\mathbb{R},L^2(\Gamma))}, \qquad s \in \mathbb{R}, \\ &\leq \|q\|_{H^{s-1/2,s-1/4}(\Gamma,\mathbb{R})}, \qquad s \geq 1/4.\end{aligned}$$

Its proof requires a definition. For each $\mu \in \mathbb{R}$, we define the operator d_t^μ: $\mathcal{S}'(\mathbb{R}^3 \times \mathbb{R}) \to \mathcal{S}'(\mathbb{R}^3 \times \mathbb{R})$ by

$$\mathcal{F}_t(d_t^\mu u)(x,\tau) = (1 + |\tau|^2)^{\mu/2} \mathcal{F}_t(u)(x,\tau), \qquad (x,\tau) \in \mathbb{R}^3 \times \mathbb{R}.$$

The mapping d_t^μ naturally extends the notion of a differentiation operator in time to distributions and for all real orders. It defines an isomorphism between a variety of Sobolev spaces. For example, it maps $H^s(\mathbb{R}, H^r(\mathbb{R}^3))$ isometrically onto $H^{s-\mu}(\mathbb{R}, H^r(\mathbb{R}^3))$ for all real numbers r, s, and μ. It is also easy to see that d_t^μ: $\mathcal{W}^{1,1/2} \to \mathcal{W}^{1,1/2}$ and d_t^μ: $\mathcal{W}^{-1,-1/2} \to \mathcal{W}^{-1,-1/2}$ are bounded for all $\mu \leq 0$.

We also introduce the analogously defined operator $\mathcal{S}'(\Gamma \times \mathbb{R}) \to \mathcal{S}'(\Gamma \times \mathbb{R})$ which we also denote by d_t^μ. Clearly, d_t^μ maps Q_s isomorphically onto $Q_{s-\mu}$ for all s,$\mu \in \mathbb{R}$. Because

$$\begin{aligned}\|d_t^\mu q\|^2_{Q^*_{-s}} &= \inf_{q_1'+q_2'=d_t^\mu q} \|q_1'\|^2_{H^{s-1/4}(\mathbb{R},L^2(\Gamma))} + \|q_2'\|^2_{H^s(\mathbb{R},H^{-1/2}(\Gamma))}, \\ &\leq \inf_{q_1+q_2=q} \|d_t^\mu q_1\|^2_{H^{s-1/4}(\mathbb{R},L^2(\Gamma))} + \|d_t^\mu q_2\|^2_{H^s(\mathbb{R},H^{-1/2}(\Gamma))}, \\ &= \inf_{q_1+q_2=q} \|q_1\|^2_{H^{s-\mu-1/4}(\mathbb{R},L^2(\Gamma))} + \|q_2\|^2_{H^{s-\mu}(\mathbb{R},H^{-1/2}(\Gamma))}, \\ &= \|q\|_{(Q_{-s+\mu})^*},\end{aligned}$$

it follows that d_t^μ also maps $(Q_{-s})^*$ into $(Q_{-s+\mu})^*$ for all s, $\mu \in \mathbb{R}$.

Proof of Theorem 5.2 Let $s \geq 0$ be fixed. We must show that $\tilde{\mathcal{K}}_1 q \in Q_s$ for all $q \in (Q_{-s})^*$. Equivalently, we will show that

$$d_t^s \tilde{\mathcal{K}}_1 q \in H^{1/2,1/4}(\Gamma, \mathbb{R}) \quad \text{for all} \quad q \in (Q_{-s})^*. \tag{5.3}$$

To show (5.3), we establish these two claims:

$$\tilde{\mathcal{K}}_1 d_t^s q \in H^{1/2,1/4}(\Gamma, \mathbb{R}) \quad \text{for all} \quad q \in (Q_{-s})^*, \tag{5.4}$$

and

$$d_t^s \tilde{\mathcal{K}}_1 q = \tilde{\mathcal{K}}_1 d_t^s q \quad \text{for all} \quad q \in \mathcal{D}(\Gamma \times \mathbb{R}). \tag{5.5}$$

By a simple density argument, (5.3) then follows from (5.4) and (5.5).

Since d_t^s: $(Q_{-s})^* \to (Q_0)^*$ and $\tilde{\mathcal{K}}_1$: $(Q_0)^* \to Q_0$ are isomorphisms, (5.4) is apparent. To prove (5.5), we note some identities. For example, by Fourier transforms, it follows that

$$d_t^{-s} \Lambda^{-1} f = \Lambda^{-1} d_t^{-s} f \quad \text{for all} \quad f \in \mathcal{S}'(\mathbb{R}^3 \times \mathbb{R}). \tag{5.6}$$

Analogously, if we let $S_\pm$ equal the subspaces (of $\mathcal{S}'(\mathbb{R}^3 \times \mathbb{R})$)

$$S_\pm = \{\theta \in \mathcal{W}^{1,1/2} \colon d_t^{\pm s} \theta \in \mathcal{W}^{1,1/2}\},$$

we can extend

$$\left(\gamma \circ d_t^{\pm s}\right) \theta = \left(d_t^{\pm s} \circ \gamma\right) \theta \quad \text{for all} \quad \theta \in \mathcal{S}(\mathbb{R}^3 \times \mathbb{R}), \tag{5.7}$$

to $\theta \in S_\pm$. Observe that $S_+ \subset \mathcal{W}^{1,1/2}$ while S_- is identical to $\mathcal{W}^{1,1/2}$. Note also that (5.7) implies

$$d_t^{-s} \gamma^* p = \gamma^* d_t^{-s} p \quad \text{for all} \quad p \in H^{-1/2,-1/4}(\Gamma, \mathbb{R}), \tag{5.8}$$

since for $p \in H^{-1/2,-1/4}(\Gamma, \mathbb{R})$ and $\theta \in \mathcal{W}^{1,1/2}$, we have

$$\begin{aligned}
\langle d_t^{-s} \gamma^* p, \theta \rangle &= \langle p, \gamma d_t^{-s} \theta \rangle \\
&= \langle p, d_t^{-s} \gamma \theta \rangle, \quad \text{by (5.7)}, \\
&= \langle \gamma^* d_t^{-s} p, \theta \rangle.
\end{aligned}$$

Now, for $q \in \mathcal{D}(\Gamma \times \mathbb{R})$, we have

$$\begin{aligned}
\gamma \, \Lambda^{-1} \, \gamma^* d_t^s q &= \gamma \, d_t^s \, d_t^{-s} \, \Lambda^{-1} \, \gamma^* d_t^s q \\
&= d_t^s \, \gamma \, \Lambda^{-1} \, d_t^{-s} \, \gamma^* d_t^s q, \quad \text{by (5.6) and (5.7)}, \\
&= d_t^s \, \gamma \, \Lambda^{-1} \, \gamma^* \, d_t^{-s} d_t^s q, \quad \text{by (5.8)}, \\
&= d_t^s \tilde{\mathcal{K}}_1 q,
\end{aligned}$$

which proves (5.5) and completes the proof of the theorem. □

We turn to proving estimate (5.2). Here, the use of localization is necessary. We shall follow the notation of section 2.2, though this time we do not index the sets. Hence, let ϕ denote a C^∞ diffeomorphism defined over some open set $O \in \mathbb{R}^3$ which maps

$$O \quad \text{onto} \quad Y = \{(y', y_3) \colon |y'| < 1,\, -1 \le y_3 \le 1\},$$

and

$$O \cap \Gamma \quad \text{onto} \quad Y_0 = \{y \in Y \colon y_3 = 0\}.$$

We again denote the inverse mapping of ϕ by ψ and let $\zeta \in \mathcal{D}(O)$ denote the cutoff function which corresponds to the set O. Finally, we recall the operators

$$\begin{aligned} \psi^*(w)(x,t) &= w(\phi(x),t) \qquad w \in H^{1,1/2}(Y,\mathbb{R}), \\ \phi^*(u)(y,t) &= u(\psi(y),t), \qquad u \in H^{1,1/2}(O,\mathbb{R}). \end{aligned}$$

Now, for $q \in H^{-1/2,-1/4}(\Gamma,\mathbb{R})$, let $u_q = \Lambda^{-1}\gamma^* q$. We need to consider the trace of u_q since this is by definition the image $\tilde{\mathcal{K}}_1 q$. Using the principles of localization, we can instead work in local coordinates and consider the trace of $w_q = \phi^*(\zeta u_q)$. Clearly,

$$\|w_q\|_{H^{1,1/2}(Y,\mathbb{R})} \le C\|q\|_{H^{-1/2,-1/4}(\Gamma,\mathbb{R})}. \tag{5.9}$$

In the next theorem, which is the key to proving (5.2), we consider the existence of the higher order spatial derivatives of w_q. We will use the multiindex notation

$$\partial^\beta = \frac{\partial^{\beta_1}}{\partial y_1}\frac{\partial^{\beta_2}}{\partial y_2}\frac{\partial^{\beta_3}}{\partial y_3},$$

where $\beta = (\beta_1, \beta_2, \beta_3) \in \mathbb{N}^3$.

Theorem 5.3 *Let m be a non-negative integer and β any multiindex with $|\beta| \le m$ and $\beta_3 = 0$. Then, there exists a positive constant C which depends on m such that*

$$\|\partial^\beta w_q\|_{H^{1,1/2}(Y,\mathbb{R})} \le C\|q\|_{H^{m-1/2,m/2-1/4}(\Gamma,\mathbb{R})}. \tag{5.10}$$

This is a standard result in the regularity theory of boundary value problems. The condition $\beta_3 = 0$ occurs since it is necessary to insure the distributional equality

$$\partial^\beta \gamma^* f = \gamma^* \partial^\beta f, \qquad f \in \mathcal{D}'(\mathbb{R}^3 \times \mathbb{R}).$$

Although its rigorous proof (based on the method of finite differences) is straightforward, it is rather tedious. Thus, we shall give the proof in Appendix A.

Although Theorem 5.3 does *not* say that w_q belongs to $L^2(\mathbb{R}, H^{m+1}(Y))$, it does supply enough information to conclude that the trace of w_q, (i.e., γw_q), belongs to $L^2(\mathbb{R}, H^{m+1/2}(\Gamma))$. This is an obvious corollary to the next lemma.

Lemma 5.4 *For any non-negative integer m, let $\theta(x) \in H^1(\mathbb{R}^3)$ satisfy*

$$\partial^\beta \theta \in H^1(\mathbb{R}^3),$$

for all multiindices β with $|\beta| \leq m$ and $\beta_3 = 0$. Then, $\gamma\theta \in H^{m+1/2}(\mathbb{R}^2)$ and satisfies

$$\|\gamma\theta\|^2_{H^{m+1/2}(\mathbb{R}^2)} \leq C(m) \sum_{\substack{|\beta|\leq m \\ \beta_3=0}} \|\partial^\beta\theta\|^2_{L^2(\mathbb{R},H^1(\mathbb{R}^3))}. \tag{5.11}$$

Proof: By Fourier transforms. The basic equation relating θ to its trace $\gamma\theta$ is

$$\mathcal{F}_{y'}\left(\gamma\theta\right)\left(\xi'\right) = \int_{-\infty}^{\infty} \hat{\theta}(\xi', \xi_3) d\xi_3, \tag{5.12}$$

where $\hat{\theta}$ denotes the Fourier transform of θ. Setting

$$k(\xi', \xi_3) = \left(1 + |\xi'|^2\right)^{m/2} \left(1 + |\xi|^2\right)^{1/2},$$

we apply the Cauchy-Schwartz inequality to (5.12) to get

$$|\mathcal{F}_{y'}(\gamma\theta)(\xi')|^2 \leq I(\xi') \int_{-\infty}^{\infty} k^2(\xi', \xi_3) |\hat{\theta}(\xi', \xi_3)|^2 d\xi_3,$$

where

$$\begin{aligned} I(\xi') &= \left(1 + |\xi'|^2\right)^{-m} \int_{-\infty}^{\infty} \frac{d\xi_3}{1 + |\xi|^2} \\ &\leq C\left(1 + |\xi'|^2\right)^{-(m+1/2)}. \end{aligned}$$

Therefore,

$$\left(1+|\xi'|^2\right)^{m+1/2}|\mathcal{F}_{y'}(\gamma\theta)(\xi')|^2 \le C\int_{-\infty}^{\infty} k^2(\xi',\xi_3)|\hat{\theta}(\xi',\xi_3)|^2 d\xi_3. \tag{5.13}$$

Integrating (5.13) over $\xi' \in \mathbb{R}^2$, we get

$$\|\gamma\theta\|^2_{L^2(\mathbb{R},H^{m+1/2}(\Gamma))} \le C\int_{\mathbb{R}^3} k^2(\xi',\xi_3)|\hat{\theta}(\xi)|^2 d\xi. \tag{5.14}$$

The lemma now follows since it is easy to verify that

$$\int_{\mathbb{R}^3} k(\xi',\xi_3)|\hat{\theta}(\xi)|^2 d\xi \le C \sum_{\substack{|\beta|\le m\\ \beta_3=0}} \|\partial^\beta w\|^2_{H^1(\mathbb{R}^3)}, \tag{5.15}$$

for some positive constant C. □

Combining Lemma 5.4 with Theorem 5.3, we have thus shown that w_q satisfies

$$\|\gamma w_q\|_{L^2(\mathbb{R},H^{m+1/2}(Y_0)} \le C(m)\|q\|_{H^{m-1/2,m-1/4}(\Gamma,\mathbb{R})}, \qquad m\in\mathbb{N}. \tag{5.16}$$

In other words, w_q satisfies the analogous estimate to (5.2) in the local coordinates. Therefore, since the function u_q in the neighborhood of Γ is simply a finite sum of functions like w_q, we conclude that $\gamma u_q = \tilde{\mathcal{K}}_1 q$ satisfies (5.2), i.e.,

$$\|\gamma u_q\|_{H^{m+1/2,m+1/4}(\Gamma,\mathbb{R})} \le C(m)\|q\|_{H^{m-1/2,m-1/4}(\Gamma,\mathbb{R})}, \qquad m\in\mathbb{N}.$$

5.2 Surjectivity

To complete the proof of Theorem 5.1, it remains to show the surjectivity of the map $\tilde{\mathcal{K}}_1$: $H^{r-1/2,r/2-1/4}(\Gamma,\mathbb{R}) \to H^{r+1/2,r/2+1/4}(\Gamma,\mathbb{R})$. As we shall see, this surjectivity follows from regularity results for the Dirichlet problem (4.13), (4.14). Over the interior set $\Omega \times \mathbb{R}$, the regularity theorem we need is well known [25, Theorem 4.2 and Theorem 5.3] and is recalled for convenience.

Theorem 5.5 *Let Ω denote any bounded, open set in $\mathbb{R}^3$. Then, for each non-negative real number r, the Dirichlet problem*

$$\begin{aligned} \frac{\partial u}{\partial t} - \Delta u = F \quad &on \quad \Omega \times \mathbb{R}, \\ u = g \quad &on \quad \Gamma \times \mathbb{R}, \end{aligned}$$

defines an isomorphism of $u \in H^{r+1,r/2+1/2}(\Omega,\mathbb{R})$ onto

$$(F,g) \in H^{r-1,r/2-1/2}(\Omega,\mathbb{R}) \times H^{r+1/2,r/2+1/4}(\Gamma,\mathbb{R}).$$

To extend this result to the Dirichlet problem posed over the exterior set, we need to introduce a new class of function spaces. We shall first define these spaces over $\mathbb{R}^3 \times \mathbb{R}$ and then by restriction over $\Omega_c \times \mathbb{R}$. For each positive integer m and $u \in \mathcal{S}(\mathbb{R}^3 \times \mathbb{R})$, set

$$\|u\|^2_{\mathcal{W}^{m,m/2}} = \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} (1 + |\xi|^2 + |\tau|)^{m-1} (|\xi|^2 + |\tau|) |\hat{u}(\xi,\tau)|^2 d\xi d\tau,$$

where $\hat{u}$ again denotes the Fourier transform of u. We now define the spaces $\mathcal{W}^{m,m/2}$ as the completions of $\mathcal{S}(\mathbb{R}^3 \times \mathbb{R})$ in these norms.

Clearly, this definition agrees with the definition of $\mathcal{W}^{1,1/2}$ when $m = 1$ and

$$\mathcal{W}^{k,k/2} \subset \mathcal{W}^{m,m/2} \quad \text{for integers} \quad 1 \le m \le k.$$

By an identical argument to the one used in section 3, we can show that $\theta(x)u(x,t) \in H^{m,m/2}$ if $\theta(x) \in \mathcal{S}(\mathbb{R}^3)$ and $u(x,t) \in \mathcal{W}^{m,m/2}$. The spaces $\mathcal{W}^{k,k/2}(\Omega^c,\mathbb{R})$ are defined as the space of restrictions to $\Omega^c \times \mathbb{R}$ of $\mathcal{W}^{m,m/2}$. The next lemma shows our motivation for introducing these spaces.

Lemma 5.6 *The heat operator Λ maps $\mathcal{W}^{m,m/2}$ onto $H^{m-2,m/2-1} \cap \mathcal{W}^{-1,-1/2}$ for all integers $m \ge 1$.*

Proof: Observe that the lemma reduces to Theorem 3.5 for $m = 1$ since

$$\mathcal{W}^{-1,-1/2} \subset H^{-1,-1/2}.$$

Thus, we assume that $m \geq 2$. There is no natural embedding between $\mathcal{W}^{-1,-1/2}$ and $H^{m-2,m/2-1}$ for such m.

Let $u \in \mathcal{S}(\mathbb{R}^3 \times \mathbb{R})$ and let $\hat{u}$ denote its Fourier transform in space and time. Clearly,

$$\mathcal{F}_{x,t}(\Lambda u)(\xi,\tau) = (i\tau + |\xi|^2)\hat{u}(\xi,\tau), \qquad (\xi,\tau) \in \mathbb{R}^3 \times \mathbb{R},$$

implies

$$\frac{1}{2}(|\xi|^2 + |\tau|)|\hat{u}(\xi,\tau)|^2 \leq \frac{|\mathcal{F}_{x,t}(\Lambda u)(\xi,\tau)|^2}{|\xi|^2 + |\tau|} \leq (|\xi|^2 + |\tau|)|\hat{u}(\xi,\tau)|^2. \quad (5.17)$$

We now multiply this equation by $(1 + |\xi|^2 + |\tau|)^{m-1}$ and then integrate over (ξ,τ). Using the definition of $\mathcal{W}^{m,m/2}$, we get

$$\frac{1}{2}\|u\|^2_{\mathcal{W}^{m,m/2}} \leq \int_{-\infty}^{\infty}\int_{\mathbb{R}^3} k_m(\xi,\tau)|\mathcal{F}_{x,t}(\Lambda u)(\xi,\tau)|^2 d\xi d\tau \leq \|u\|^2_{\mathcal{W}^{m,m/2}}$$

where

$$k_m(\xi,\tau) = \frac{(1 + |\xi|^2 + |\tau|)^{m-1}}{|\xi|^2 + |\tau|}.$$

To complete the proof, it suffices to show that

$$f \mapsto \left\{\int_{-\infty}^{\infty}\int_{\mathbb{R}^3} k_m(\xi,\tau)|\mathcal{F}_{x,t}(f)(\xi,\tau)|^2 d\xi d\tau\right\}^{1/2}, \quad (5.18)$$

defines a norm over $H^{m-2,m/2-1} \cap \mathcal{W}^{-1,-1/2}$ which is equivalent to

$$f \mapsto \left\{\|f\|^2_{H^{m-2,m/2-1}} + \|f\|^2_{\mathcal{W}^{-1,-1/2}}\right\}^{1/2}.$$

But, since $m \geq 2$ and

$$\|f\|^2_{\mathcal{W}^{-1,-1/2}} = \int_{-\infty}^{\infty}\int_{\mathbb{R}^3}(\,|\xi|^2 + |\tau|\,)^{-1}|\hat{f}(\xi,\tau)|^2 d\xi d\tau,$$

and

$$\|f\|^2_{H^{m-2,m/2-1}} = \int_{-\infty}^{\infty}\int_{\mathbb{R}^3}(\,1 + |\xi|^2 + |\tau|\,)^{m-2}|\hat{f}(\xi,\tau)|^2 d\xi d\tau.$$

this is an obvious corollary of the binomial theorem. □

Remark: Intuitively, the reason that the $\mathcal{W}^{-1,-1/2}$ term must be explicitly included above is because the space $\mathcal{W}^{m,m/2}$ imprecisely describes the behavior of solutions at infinity.

Using Lemma 5.6, Theorem 5.5 and the principles of localization, the next result can be shown. As before, the proof is tedious but straightforward. Thus, we give the proof in appendix B.

Theorem 5.7 *Let $g \in H^{m+1/2,m/2+1/4}(\Gamma,\mathbb{R})$ for some integer $m \geq 0$. Then, the solution $u \in \mathcal{W}^{1,1/2}$ to*

$$\frac{\partial u}{\partial t} - \Delta u = 0 \quad on \quad \mathbb{R}^3 \backslash \Gamma \times \mathbb{R}, \tag{5.19}$$

$$u = g \quad on \quad \Gamma \times \mathbb{R}, \tag{5.20}$$

satisfies

$$u|_\Omega \in H^{m+1,m/2+1/2}(\Omega,\mathbb{R}) \quad and \quad u|_{\Omega^c} \in \mathcal{W}^{m+1,m/2+1/2}(\Omega^c,\mathbb{R}).$$

From Theorem 5.7, the surjectivity of the mapping $\tilde{\mathcal{K}}_1$ is easily deduced.

Theorem 5.8 *For all $r \geq 0$, the operator $\tilde{\mathcal{K}}_1$ maps $H^{r-1/2,r/2-1/4}(\Gamma,\mathbb{R})$ onto $H^{r+1/2,r/2+1/4}(\Gamma,\mathbb{R})$.*

Proof: Suppose that r equals a non-negative integer m. Given $g \in H^{m+1/2,m/2+1/4}(\Gamma,\mathbb{R})$, let u denote the solution to (5.19-5.20). Set

$$q = \frac{\partial u_+}{\partial \mathbf{n}} - \frac{\partial u_-}{\partial \mathbf{n}}.$$

From Theorem 5.7 and a trace theorem (cf., the remark after Theorem 2.3), it follows that each of the normal derivatives satisfies the inclusions

$$\frac{\partial u_+}{\partial \mathbf{n}} \in H^{m-1/2,m/2-1/4}(\Gamma,\mathbb{R})$$

and

$$\frac{\partial u_-}{\partial \mathbf{n}} \in H^{m-1/2,m/2-1/4}(\Gamma,\mathbb{R}).$$

Hence, we have $q \in H^{m-1/2,m/2-1/4}(\Gamma,\mathbb{R})$ and by (4.18) and (4.20), $\tilde{\mathcal{K}}_1 q = g$. The surjectivity for all non-negative values of r follows by interpolation. □

With the proof of Theorem 5.1 complete, we apply it to consider the regularity of the operator $\mathcal{K}_1$ in the next theorem.

Theorem 5.9 *The single layer potential $\mathcal{K}_1$ maps*

$$H_{00}^{r-1/2,r/2-1/4}(\Gamma,\mathbb{R}_+) \quad onto \quad H_{00}^{r+1/2,r/2+1/4}(\Gamma,\mathbb{R}_+), \quad for\ all \quad r \geq 0.$$

To prove this result, we need a few preliminary results

Theorem 5.10 *The heat operator Λ maps $\{u \in \mathcal{W}^{1,1/2} : u = 0 \quad t < 0\}$ isomorphically onto $\{F \in \mathcal{W}^{-1,-1/2} : F = 0,\ t < 0\}$.*

A proof of this result is given in [22, pages 403–408]. We state a useful corollary to this result.

Corollary 5.11 *Let $g \in H^{1/2,1/4}(\Gamma,\mathbb{R})$ be such that $g = 0$ almost everywhere for $t < 0$. Then, the unique solution $u \in \mathcal{W}^{1,1/2}$ to the Dirichlet problem (4.13–4.14) equals zero for $t < 0$.*

Proof: Let $g \in H^{1/2,1/4}(\Gamma,\mathbb{R})$ with $g = 0$ for $t < 0$ be given. The crux of the matter is to note there exists a trace extension $\mathcal{E}g \in \mathcal{W}^{1,1/2}$ of g which satisfies $\mathcal{E}g = 0$ for $t < 0$. It is not to hard to see that there are many extensions which satisfy this. For example, in the local coordinates, the multiplication map

$$g(y') \to \rho(y_n)g(y'), \qquad (y', y_3) \in \mathbb{R}^3,$$

where $\rho \in \mathcal{D}(\mathbb{R})$ with $\rho(0) = 1$ is clearly such an extension, as is the extension defined by it using the procedure described in section 2.2.

Now, recall how the Dirichlet solution u is determined. We have that $u = \mathcal{E}g + u_\Gamma$ where $u_\Gamma \in \mathcal{W}_\Gamma^{1,1/2}$ is determined by

$$\langle \Lambda u_\Gamma, v\rangle = -\langle \Lambda \mathcal{E}g, v\rangle \quad \text{for all} \quad v \in \mathcal{W}_\Gamma^{1,1/2}.$$

Since $\mathcal{E}g$ vanishes for $t < 0$, it follows that $\Lambda\mathcal{E}g$ also vanishes for $t < 0$ and by Theorem 5.10, so does u_Γ. □

In the next theorem, we show that $\tilde{\mathcal{K}}_1$ satisfies properties quite analogous to the ones in Theorem 5.10 satisfied by the heat operator Λ.

Theorem 5.12 *The operator $\tilde{\mathcal{K}}_1$ maps $\{q \in H^{-1/2,-1/4}(\Gamma,\mathbb{R}) : q = 0,\ t < 0\}$ isomorphically onto $\{g \in H^{1/2,1/4}(\Gamma,\mathbb{R}) : g = 0,\ t < 0.\}$*

Proof: Let $q \in H^{-1/2,-1/4}(\Gamma, \mathbb{R})$ with support in $t > 0$ be given. Since the trace operator γ is purely a spatial operator, it is easy to see that $\gamma^* q \in \mathcal{W}^{-1,-1/2}$ also has support in $t > 0$. Hence, by Theorem 5.10, the function $\Lambda^{-1}\gamma^* q \in \mathcal{W}^{1,1/2}$ has support in $t > 0$. Thus, so does $\gamma\Lambda^{-1}\gamma^* q = \tilde{\mathcal{K}}_1 q$.

Conversely, let $g \in H^{1/2,1/4}(\Gamma, \mathbb{R})$ with support in $t > 0$ be given. It suffices to show that

$$\langle \tilde{\mathcal{K}}_1^{-1} g, \rho \rangle = 0,$$

for any arbitrarily fixed $\rho \in H^{1/2,1/4}(\gamma, \mathbb{R})$ which equals zero for $t > 0$. To show this, let u_g denote the unique solution to the Dirichlet problem (4.13–4.14). By Corollary 5.11, u_g vanishes for $t < 0$. Now, recall our discussion at the end of section 4. In this section, we showed that $\tilde{\mathcal{K}}_1^{-1} g$ agrees with the jump in the normal derivatives of u_g across the boundary Γ. Clearly, these normal derivatives and thus q vanish for t <0. □

It is now a simple matter to prove Theorem 5.9. Given any q in the Sobolev space $H_{00}^{r-1/2,r/2-1/4}(\Gamma, \mathbb{R}_+)$, it follows that its zero extension $R_+^* q$ satisfies

$$R_+^* q \in H^{r-1/2,r/2-1/4}(\Gamma, \mathbb{R}),$$

and vanishes for $t < 0$. Hence, $\tilde{\mathcal{K}}_1 R_+^* q$ belongs to $H^{r+1/2,r/2+1/4}(\Gamma, \mathbb{R})$ and vanishes for $t < 0$. By definition, its restriction to $\mathbb{R}_+$, namely $R_+ \tilde{\mathcal{K}}_1 R_+^* q = \mathcal{K}_1 q$ belongs to $H_{00}^{r+1/2,r/2+1/4}(\Gamma, \mathbb{R}_+)$. The other direction follows with equal ease.

6 Galerkin Discretization of the First Kind Boundary Integral Equation

In the remaining sections, we discuss the numerical solution of the first kind boundary integral equation

$$\mathcal{K}_1 q(x,t) := \int_0^t \int_\Gamma K(x-y,t-s)q(y,s)dyds = F(x,t), \qquad (x,t) \in \Gamma \times \mathbb{R}_+ \tag{6.1}$$

by Galerkin methods. The coercivity of single layer potential $\mathcal{K}_1$ implies the quasioptimality of Galerkin approximations. That is, if Q_h denotes any closed subspace of $H^{-1/2,-1/4}(\Gamma,\mathbb{R}_+)$ and $q_h \in Q_h$ the unique solution to the Galerkin equations

$$\langle p, \mathcal{K}_1 q_h \rangle = \langle p, F \rangle \quad \text{for all} \quad p \in Q_h, \tag{6.2}$$

then

$$\|q - q_h\|_{H^{-1/2,-1/4}(\Gamma,\mathbb{R}_+)} \leq C \inf_{q_h' \in Q_h} \|q - q_h'\|_{H^{-1/2,-1/4}(\Gamma,\mathbb{R}_+)}. \tag{6.3}$$

In section 6.1, we describe a standard class of tensor product spaces $Q_h^{d_x,d_t} = V_h^{d_t} \otimes X_h^{d_x}$ which are based on polynomials of degree d_t in time and polynomials of degree d_x in space. In sections 6.2 and 6.3, we consider the implementation of the Galerkin method using the space V_h^0 of piecewise constants in time. In section 6.4, we consider the Galerkin equations when higher order polynomials are used in time.

6.1 Construction of the Trial Space

Let h_t denote the stepsize of a uniform partition of $\mathbb{R}_+$. We define V_h^0 as the space of piecewise constant functions subordinate to this mesh. This space is conveniently described as the span of the basis functions

$$\chi_k(t) = \begin{cases} 1, & (k-1)h_t < t < kh_t, \\ 0, & \text{otherwise} \end{cases}, \qquad k = 1, 2, \dots .$$

The L^2 projection operator P_t: $L^2(\mathbb{R}_+) \to V_h^0$ is defined by

$$P_t u(t) = \sum_{k=1}^{\infty} \frac{1}{h_t} \langle u, \chi_k \rangle \chi_k(t). \tag{6.4}$$

Note that P_t extends to an operator on $H^{s_2}(\mathbb{R}_+)$ for $s_2 > -1/2$, since the functions χ_k belong to $H^{s_1}(\mathbb{R}_+)$ for $s < 1/2$. Analogously, for any $d_t > 0$, we define $V_h^{d_t}$ to be the space of piecewise, *discontinuous* polynomials of degree d_t.

A description of X_h is more involved. We assume that Γ can be divided into M closed subsets Γ_m such that

$$\Gamma = \bigcup_{m=1}^{M} \Gamma_m, \quad \text{and} \quad \Gamma_{m_1} \cap \Gamma_{m_2} \text{ is a curve, a point, or empty for } m_1 \neq m_2. \tag{6.5}$$

We assume that each piece Γ_m can be smoothly mapped in a 1 to 1 fashion onto the unit square

$$\hat{S}_1 = \{(\sigma_1, \sigma_2) \colon 0 \leq \sigma_1,\, \sigma_2 \leq 1\}.$$

We denote the mappings $\Gamma_m \to \hat{S}_1$ by Φ_m and assume that each Φ_m is the restriction to Γ of some C^∞ diffeomorphism which maps an open $\mathbb{R}^3$ neighborhood of Γ_m onto some open $\mathbb{R}^3$ neighborhood of $\hat{S}_1$. Thus, it makes sense to define the inverse mappings $\Psi_m \colon \hat{S}_1 \to \Gamma_m$. Without loss of generality, we will suppose that the various Jacobians of Φ_m and Ψ_m are strictly positive.

The mappings Ψ_m will be used to define interpolation over Γ. For this purpose, it is necessary that they piece together correctly. Precisely, we require that $\Phi_{m_2} \circ \Psi_{m_1} \colon \Phi_{m_1}(\Gamma_{m_1} \cap \Gamma_{m_2}) \to \Phi_{m_2}(\Gamma_{m_1} \cap \Gamma_{m_2})$ is an isometry if $\Gamma_{m_1} \cap \Gamma_{m_2} \neq \emptyset$.

Remark: Our assumptions on Γ are fairly general and apply to most surfaces encountered in practice. For example, they apply to any convex surface. The same sort of assumptions have been made by [2], [28], and [41]. An important case which does not require an elaborate boundary decomposition is when Γ is a polygonal surface. Our analysis, however, does not strictly apply to this case since regularity assumptions which are justified on smooth surfaces by our regularity results fail to hold on polygonal surfaces.

For each m, let $\hat{\mathcal{R}}_h^m$ denote a regular triangulation of $\hat{S}_1$ by rectangular elements. For each rectangle $\hat{r} \in \hat{\mathcal{R}}$, define curvilinear rectangles r by

$$r = \{x \in \mathbb{R}_3 \colon \Phi_m(x) \in \hat{r}\}.$$

Let R_h^m be the set of all such rectangles r as $\hat{r}$ varies in $\hat{\mathcal{R}}_h^m$. The union

$$\mathcal{R}_h = \bigcup_m R_h^m,$$

is a triangulation of the surface Γ by curvilinear rectangles. We set

$$h_x = \max_{r \in \mathcal{R}_h} \text{diam r}.$$

The spaces $X_h^{d_x}$ are now defined as images of spaces of piecewise polynomials defined over the triangulation $\hat{\mathcal{R}}_h$. For any non-negative integer d_x and any set $\hat{r} \in \hat{\mathcal{R}}_h$, let $\hat{\mathcal{P}}(\hat{r}, d_x)$ denote the tensor product space of d_x degree polynomials in each variable. We then define

$$X_h^{d_x} = \{q_h \in L^2(\Gamma): \quad q_h|_r \circ \Psi_m \in \hat{\mathcal{P}}(\hat{r}, d_x); \quad \text{for all } r \in \mathcal{R}_h^m,\ m = 1, \ldots, M\}.$$

Basis functions for $X_h^{d_x}$ are defined as the images of standard polynomial basis functions. Let $\hat{\nu}^j(\sigma_1, \sigma_2)$ (for $j = 1$ to $(d_x + 1)^2$) denote the basis functions of $\mathcal{P}(\hat{S}_1)$. (For example, if $d_x = 1$, these four functions are

$$\begin{array}{rclrcl} \hat{\nu}^1(\sigma_1, \sigma_2) & = & \sigma_1\sigma_2, & \hat{\nu}^3(\sigma_1, \sigma_2) & = & (1 - \sigma_1)\sigma_2, \\ \hat{\nu}^2(\sigma_1, \sigma_2) & = & \sigma_1(1 - \sigma_2), & \hat{\nu}^4(\sigma_1, \sigma_2) & = & (1 - \sigma_1)(1 - \sigma_2). \end{array} \Bigg)$$

Given $r \in R_h^m$, let $\hat{r} = \Psi_m(r)$ and denote by $\hat{\Theta}$ the dilation from $\hat{r}$ onto $\hat{S}_1$. Then, for each $r \in R_h^m$, it is easy to check that the set of functions

$$\nu_r^j = \hat{\nu}^j \circ \hat{\Theta} \circ \Phi_m, \qquad j = 1, \ldots, (d_x + 1)^2, \tag{6.6}$$

defines a basis for $X_h|_r$. Thus, the collection of functions

$$\{\nu_r^j, \qquad j = 1, \ldots (d_x + 1)^2,\ r \in \mathcal{R}_h\}$$

forms a basis for $X_h^{d_x}$ In formulating the Galerkin equations, it will be convenient to reindex the basis functions in $X_h^{d_x}$ and to denote them by $\{\nu_\alpha(x)\}$. Note that the doubly indexed set $\{\nu_\alpha(x)\chi_n(t)\}$ forms a basis of $Q_h^{d_t, d_x}$.

6.2 Implementation

The Galerkin equations (6.2) in the subspace $Q_h^{1,0}$ are the linear system

$$\langle \nu_\alpha \chi_n, \mathcal{K}_1 q_h \rangle = \langle \nu_\alpha \chi_n, F \rangle, \qquad \alpha = 1, 2, \dots, N_x,\ n \in \mathbb{Z}_+, \tag{6.7}$$

where N_x denotes the dimension of X_h. Since the Galerkin solution q_h belongs to $Q_h^{1,0}$, we can expand it in terms of the Q_h basis functions as

$$q_h(x,t) = \sum_{k=1}^{\infty} \sum_{\beta=1}^{N_x} q_{\beta k} \nu_\beta(x) \chi_k(t).$$

Substituting this expansion into (6.7), we get

$$\sum_{k=1}^{\infty} \sum_{\beta=1}^{N_x} \langle \nu_\alpha \chi_n, \mathcal{K}_1 \nu_\beta \chi_k \rangle q_{\beta k} = \langle \nu_\alpha \chi_n, F \rangle, \qquad \alpha = 1, \dots, N_x,\ n \in \mathbb{Z}_+. \tag{6.8}$$

Since (6.8) is indexed by four integers, its solution requires some ordering or partitioning of the unknowns. For each positive integer n, we define vectors $\vec{q}_n$ of length N_x by

$$\vec{q}_n = \begin{pmatrix} q_{1,n} \\ q_{2,n} \\ \vdots \\ q_{N_x,n} \end{pmatrix}, \qquad n \in \mathbb{Z}_+.$$

Similarly, we let $\vec{F}_n$ denote vectors of length N_x whose components are given by

$$(\vec{F}_n)_\alpha = \int_\Gamma \nu_\alpha(x) \int_{(n-1)h_t}^{nh_t} F(x,t)\,dt\,dx, \qquad 1 \le \alpha \le N_x,\ n \in \mathbb{Z}_+.$$

Finally, we define square matrices, G_{nk}, of order N_x for each $n, k \in \mathbb{Z}_+$ by

$$(G_{n,k})_{\alpha,\beta} = \int_\Gamma \int_\Gamma \nu_\alpha(x) g_{n,k}(x - x') \nu_\beta(x')\,dx'\,dx, \tag{6.9}$$

where for any $x \in \mathbb{R}^3$, $g_{n,k}(x)$ denotes

$$g_{n,k}(x) = \int_{(n-1)h_t}^{nh_t} \int_{(k-1)h_t}^{kh_t} K(x, t - t')\,dt'\,dt. \tag{6.10}$$

With this notation, the Galerkin system (6.8) partitions in the form

$$\begin{pmatrix} G_{11} & G_{12} & G_{13} & \dots \\ G_{21} & G_{22} & G_{23} & \dots \\ G_{31} & G_{32} & G_{33} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \vec{q}_1 \\ \vec{q}_2 \\ \vec{q}_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} \vec{F}_1 \\ \vec{F}_2 \\ \vec{F}_3 \\ \vdots \end{pmatrix}. \tag{6.11}$$

Now

$$\langle \chi_n, \mathcal{K}_1 \chi_k \rangle = 0 \quad \text{for} \quad n > k, \tag{6.12}$$

so $G_{nk} = 0$ for $n > k$. Thus, (6.11) has the block lower triangular form

$$\begin{pmatrix} G_{11} & 0 & 0 & \dots \\ G_{21} & G_{22} & 0 & \dots \\ G_{31} & G_{32} & G_{33} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \vec{q}_1 \\ \vec{q}_2 \\ \vec{q}_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} \vec{F}_1 \\ \vec{F}_2 \\ \vec{F}_3 \\ \vdots \end{pmatrix}. \tag{6.13}$$

We point out that this lower triangular form occurs because the supports $\chi_{k_1}(t)$ and $\chi_{k_2}(t)$ for different values of k_1 and k_2 are disjoint. Therefore, a lower triangular form would still result if one defined V_h using a non-uniform mesh.

Expanding (6.13) out, we get the set of order N_x linear systems

$$\sum_{k=1}^{n} G_{nk} \vec{q}_k = \vec{F}_n, \qquad n = 1, 2, \dots, N_x,$$

or

$$G_{nn} \vec{q}_n = \vec{F}_n - \sum_{k=1}^{n-1} G_{nk} \vec{q}_k, \qquad n = 1, 2, \dots, N_t \tag{6.14}$$

Observe how (6.14) successively determines the solution at each timestep as the solution to a linear system of order N_x. We remark that the invertibility of the matrices G_{nn} is a direct consequence of the coercivity of the operator $\mathcal{K}_1$. (Consider the pairing $\langle p, \mathcal{K}_1 p \rangle$ for any $p \in Q_h^{1,0}$ which is support in the interval $(n-1)h_t < t < nh_t$.)

For constant time steps, an important savings occurs since the matrices $G_{n,k}$ satisfy

$$G_{n_2 k_2} = G_{n_1 k_1} \quad \text{when} \quad n_2 - k_2 = n_1 - k_1. \tag{6.15}$$

Equation (6.15) is very important since the computation of the solution $\vec{q}_n$ at each additional time step now only requires the generation of one new matrix. Moreover, the only matrix that needs to be inverted in solving (6.14) is G_{11}. Thus, a standard matrix decomposition algorithm, such as the Cholesky decomposition, may be applied to this matrix at the beginning of an algorithm and subsequently stored in G_{11}. These savings are so significant that non-constant meshes are never used in computations.

To implement (6.14), it remains to compute the matrix elements. Fortunately, the time integrations required to evaluate $g_{nk}(x)$ can be done analytically. To see this, let $l = n - k$ in (6.10) and scale the variables of integration to get

$$\begin{aligned} g_{n,k}(x) &= h_t^2 \int_0^1 dt \int_0^1 K(x, h_t(t - t' + l))dt' \\ &= h_t^2 \int_0^1 dt \int_{t+(l-1)}^{t+l} K(x, h_t s)ds. \end{aligned}$$

Changing the order of integration, we get

$$\begin{aligned} g_{n,k}(x) &= h_t^2 \int_{l-1}^{l} \int_0^{s-l-1} K(x, sh_t)dtds \\ &\quad + h_t^2 \int_l^{l+1} \int_{s-l}^{1} K(x, sh_t)dtds. \end{aligned} \tag{6.16}$$

In this equation the first integral is to be considered zero when $l = 0$. Performing the integration over t, we have

$$\begin{aligned} g_{n,k}(x) &= h_t^2 \int_{l-1}^{l} K(x, sh_t)\,(s - l + 1)\,ds \\ &\quad + h_t^2 \int_l^{l+1} K(x, sh_t)\,(l + 1 - s)\,ds. \end{aligned}$$

Since

$$\int_0^t s^{-(\kappa+1)} e^{-r^2/4s} ds = \frac{4^\kappa}{r^{2\kappa}} \int_{r^2/4t}^{\infty} \sigma^{\kappa-1} e^{-\sigma} d\sigma, \tag{6.17}$$

for all real κ, we see that $g_{n,k}(x)$ may be expressed in terms of the family of functions

$$\Upsilon(\kappa, z) = \int_z^\infty e^{-\sigma} \sigma^{\kappa-1} d\sigma, \qquad \kappa \geq 0. \tag{6.18}$$

Setting

$$a_l = \frac{x^2}{4lh_t}, \qquad l \in \mathbb{Z}_+, \tag{6.19}$$

and working out the details, we get

$$\begin{aligned}\frac{4\pi^{3/2}}{h_t} g_{n,k}(x) &= \frac{1}{r}(l + 1 + \sqrt{a_1})\left(\Upsilon(a_{1+1}, 1/2) - \Upsilon(a_l, 1/2)\right) \\ &\quad - \frac{1}{r}(l - 1 + \sqrt{a_1})\left[\Upsilon(a_l, 1/2) - \Upsilon(a_{l-1}, 1/2)\right] \\ &\quad + \left[2le^{-a_l} - (l+1)e^{-a_{l+1}} - (l-1)e^{-a_{l-1}}\right], \end{aligned} \tag{6.20}$$

for all values of (n, k) such that $l \geq 1$. (When $l = 1$, one uses the limits

$$\lim_{z \to 0} z e^{-1/z} = 0 \quad \text{and} \quad \lim_{z \to \infty} \Upsilon(z) = 0.)$$

The appropriate expression for $g_{n,n}(x)$ is

$$g_{n,n}(x) = \frac{h_t}{4\pi^{3/2}}\left((1 + \sqrt{a_1})\frac{1}{r}\Upsilon(a_1, 1/2) - e^{-a_1}\right).$$

The functions Υ are known as the complementary incomplete gamma functions and are customarily denoted by $\Gamma(\kappa, z)$ (we avoid this notation because it conflicts with our notation for the spatial surface). They have been extensively studied and tabulated. For all positive values of κ, they tend to a finite limit as z tends to zero (namely, the value of the complete gamma function). When z tends to infinity, these functions decrease exponentially to zero. More precisely, we have

$$\Upsilon(\kappa, z) \sim e^{-z} z^{\kappa - 1} \quad \text{as} \quad z \to \infty, \qquad \kappa > 0.$$

For our purposes, we only need $\Upsilon(\kappa, z)$ for $\kappa = j + 1/2$ where $j \in \mathbb{N}$. Since we have the recurrence relation

$$\Upsilon(\kappa + 1, z) = e^{-z} z^{\kappa} + \Upsilon(\kappa, z),$$

(derived by integration by parts), the only non-trivial evaluation required is determining the value of $\Upsilon(1/2, z)$. Further, the change of variable $\sigma = \omega^2$ shows that

$$\Upsilon(1/2, z) = 2\int_{\sqrt{z}}^{\infty} e^{-\omega^2} d\omega, \qquad z \in \mathbb{R}_+.$$

Thus, $\Upsilon(1/2, z)$ is related to the complementary error function whose values are extensively available.

Unfortunately, the computation of the spatial integrals over the boundary

$$(G_{n,k})_{\alpha,\beta} = \int_\Gamma \int_\Gamma \nu_\alpha(x) g_{n,k}(x - x')\nu_\beta(x')dxdx',$$

must be done numerically. Based on the behavior of the gamma functions, it follows that $g_{n,n}(x)$ has a $|x|^{-1}$ type singularity as x tends to the origin. (This singularity is completely analogous to the behavior of the electrostatic single layer potential near the origin.) Accordingly, a special type of integration rule must be used to compute the diagonal elements of $G_{n,n}$. Fortunately, this issue has received considerable attention in the literature. For example, see [15] and [41]. Since the functions $g_{n,k}(x)$ for $k < n$ are smooth, a regular Gaussian integration rule can be used to compute the matrices $G_{n,k}$.

6.3 Application to the Direct Integral Equation

Throughout this paper, our motivating example of the single layer operator equation $\mathcal{K}_1 q = F$ has been the direct integral equation. To recall, the forcing function F of this equation is given by

$$\begin{aligned} F(x,t) &= \frac{1}{2}g(x,t) + \int_0^t \int_\Gamma \frac{\partial K}{\partial \mathbf{n}}(x-y, t-s)g(y,s)dyds \\ &\quad + \int_\Omega K(x-x', t)f(x')dx', \qquad (x,t) \in \Gamma \times \mathbb{R}_+, \quad (6.21) \end{aligned}$$

where $g \in H^{1/2,1/4}(\Gamma, \mathbb{R}_+)$ and $f \in L^2(\Omega)$ are known. (They represent the prescribed Dirichlet and Cauchy data of the boundary value heat equation under consideration.) In this section, we describe an approximation F_h to F and address the evaluation of integrals required to apply the Galerkin method to this equation.

Define

$$\mathcal{M}f = \int_\Omega K(x - x', t)f(x')dx', \qquad x \in \Omega,\ t > 0,$$

and recall the notation

$$\mathcal{K}_2 g(x,t) = \int_0^t \int_\Gamma \frac{\partial K}{\partial \mathbf{n}}(x-y, t-s)g(y,s)dyds, \qquad x \in \Gamma,\ t > 0.$$

Thus, $F = (1/2 + \mathcal{K}_2)g + \mathcal{M}f$. Our approximation of this function will be based on polynomial interpolations of g and f. Since interpolation over the set $\Gamma \times \mathbb{R}_+$ has already been discussed, we concentrate on interpolation over the set Ω.

Unfortunately, this necessitates some sort of triangulation of Ω. To include non-polygonal sets Ω in our discussion, we shall use an isoparametric triangulation [9]. We let $\mathcal{T}$ represent such a triangulation (regular and quasi-uniform) of Ω and set

$$\Omega_{\mathcal{T}} = \bigcup_{T \in \mathcal{T}} T \quad \text{and} \quad h_{\mathcal{T}} = \sup_{T \in \mathcal{T}} \operatorname{diam}(T).$$

For simplicity, we will assume that the open set $\Omega_{\mathcal{T}}$ is contained in Ω. This assumption is often true in practice and is simply made to allow us to properly consider f over $\Omega_{\mathcal{T}}$.

Since $\Omega_{\mathcal{T}}$ is a polygonal set, it makes sense to define the piecewise linear interpolant $\Pi_1 f$ of f with respect to the triangulation $\mathcal{T}$. Because $\Pi_1 f$ is defined only over $\Omega_{\mathcal{T}}$, it is necessary to define the operator

$$\mathcal{M}_{\mathcal{T}} f = \int_{\Omega_{\mathcal{T}}} K(x - x', t) f(x') dx', \qquad x \in \Omega,\, t > 0,$$

and approximate the domain term by $\mathcal{M}_{\mathcal{T}} \Pi_1 f$.

Analogously, our approximation of the double layer term $(1/2 + \mathcal{K}_2)g$ is based on replacing g by its tensor product interpolant $P_x P_t g \in Q_h^{1,0}$. Note that the use of linear (or higher) degree polynomials in space is required to ensure the inclusion $P_x P_t g \in H^{1/2,1/4}(\Gamma, \mathbb{R}_+)$. Combined with our remarks above, we define the function F_h

$$F_h(x,t) = (1/2 + \mathcal{K}_2) P_x^1 P_t^1 g(x,t) + \mathcal{M}_{\mathcal{T}} \Pi_1 f(x,t).$$

To implement the Galerkin method, we must evaluate the integrals

$$\int_\Gamma \nu_\alpha(x) \int_{(n-1)h_t}^{nh_t} F(x,t) dt dx, \qquad 1 \leq \alpha \leq N_x,\, n \in \mathbb{Z}_+,$$

The integral in time may be taken exactly, with the spatial integrals requiring numerical quadrature. We refer the reader to [5],[34] for details. Incorporating these approximations into the Galerkin equations (6.14), we get a discretized system which in matrix form looks like

$$\sum_{k=1}^{n} G_{n,k} \vec{q}_k = \sum_{k=1}^{n} L_{n,k} \vec{g}_k + M_n \vec{f}, \qquad n \in \mathbb{Z}_+, \tag{6.22}$$

where (for each $n, z \in \mathbb{Z}_+$), $G_{n,k}$ and $L_{n,k}$ denote square matrices of order N_x and $M_n \in \mathbb{R}^{N_\mathcal{T} \times N_x}$.

An immediate observation from (6.22) is the great deal of storage this method requires. (We point out that the excessive storage is *not* due to the use of a Galerkin method.) Shortly, we mention an alternate formulation of the boundary element equations which minimizes the needed storage.

Often, the goal of computations is not the computation of the fluxes q_n, but in using these values to obtain estimates of the solution $u(x,t)$ to the given boundary value problem. Such estimates are obtained from the representation formula (1.4). With q_h denoting the Galerkin solution, $\mathcal{M}_\mathcal{T}\Pi_1 f$, and $P_x P_t g$ as above, the function

$$u_h(x,t) = \int_0^t \int_\Gamma [K(x-y,t-s)q_h(y,s) - \frac{\partial K}{\partial \mathbf{n}}(x-y,t-s)P_x P_t g(y,s)]dyds + \int_\Omega K(x-x',t)\Pi_1 f(x')dx, \quad (x,t) \in \mathbb{R}^3 \times \mathbb{R}_+. \quad (6.23)$$

defines a natural approximation to the solution u. In practice, u_h is only evaluated at a finite number of points. A typical procedure is to evaluate u_h at the node points of $\Omega_\mathcal{T} \times \mathbb{R}_+$, with values of u_h at other points being approximated by interpolation.

An alternate approach to solving the boundary element equations is based on (6.23) and the time invariance of the heat equation. Specifically, let $\vec{q}_1$ denote the solution to the Galerkin equations. As discussed above, this value determines the approximate values of $u(x_\alpha, t_1)$ in the interior. We now consider these values to represent the initial values in (6.23) . This procedure greatly reduces storage since it only requires the solution of the linear system

$$G_{11}\vec{q}_n = L_{11}\vec{g}_n + M_n \vec{u}_{n-1}, \quad (6.24)$$

along with an interior equation of the form

$$\vec{u}_n = \sum_{k=n-1}^{n} \left(G^\Omega_{n,k}\vec{q}_k + L^\Omega_{n,k}\vec{g}_k \right) + M^\Omega_1 \vec{f}_{n-1}, \qquad n \geq 1.$$

Note how this approach couples the computation of the boundary values $\partial u/\partial \mathbf{n}$ and the interior values of u. The drawback this has it that it requires a triangulation of the domain Ω even if the initial data f is zero. It therefore would not be used in this case. (Also if f satisfies Laplace's equation. See [5] for details.) On the other hand, for problems where f is non-zero and interior values are desired, this method would seem to be worthy of some further attention. To date, the most complete investigation of this method has been given in [30].

6.4 Higher Order Methods in Time

We consider the Galerkin equations when we base V_h on piecewise linear polynomials. (Higher degree polynomials can be handled similarly.) First, we treat the case of *discontinuous* polynomials. Over the reference interval $(0,1)$, we set

$$\hat{\rho}^1(\sigma) = 1 - \sigma \quad \text{and} \quad \hat{\rho}^2(\sigma) = \sigma,$$

and then introduce the basis functions

$$\rho_k^1(t) = \hat{\rho}^1(\frac{t}{h_t} + 1 - k) \quad \text{and} \quad \rho_k^2(t) = \hat{\rho}^2(\frac{t}{h_t} + 1 - k), \qquad k \in \mathbb{Z}_+.$$

We now write each $q \in Q_h$ as

$$q(x,t) = \sum_{\beta=1}^{N_x} \sum_{k=1}^{\infty} \nu_\beta(x) \left(q_{\beta k}^1 \rho_k^1(t) + q_{\beta k}^2 \rho_k^2(t) \right), \tag{6.25}$$

and introduce over each subinterval the pair of vectors

$$\vec{q}_n^{\,1} = \begin{pmatrix} q_{1n}^1 \\ q_{2n}^1 \\ \vdots \\ q_{N_x n}^1 \end{pmatrix} \quad \text{and} \quad \vec{q}_n^{\,2} = \begin{pmatrix} q_{1n}^2 \\ q_{2n}^2 \\ \vdots \\ q_{N_x n}^2 \end{pmatrix}, \quad n \in \mathbb{N}. \tag{6.26}$$

To express the Galerkin equations in Q_h, let κ_1 and κ_2 denote numbers which either equal one or two. Then, for each $l \in \mathbb{N}$, we define matrices $G_l^{\kappa_1,\kappa_2}$ each of order N_x by

$$\begin{aligned} (G_l^{\kappa_1,\kappa_2})_{\alpha,\beta} &= \int_\Gamma \int_\Gamma \nu_\alpha(x) \nu_\beta(x') g_l^{\kappa_1,\kappa_2}(x - x') dx dx' \\ &\qquad 1 \le \alpha, \beta \le N_x, \end{aligned} \tag{6.27}$$

where

$$g_l^{\kappa_1,\kappa_2}(x) = \int_{lh_t}^{(l+1)h_t} \int_0^t \rho_{l+1}^{\kappa_1}(t) K(x, t-t') \rho_1^{\kappa_2}(t') dt' dt, \qquad x \in \mathbb{R}^3.$$

By defining matrices of order $2N_x$ as

$$\mathcal{G}_l := \begin{pmatrix} G_l^{11} & G_l^{12} \\ G_l^{21} & G_l^{22} \end{pmatrix}, \qquad l \in \mathbb{N}, \tag{6.28}$$

we can write the Galerkin equations in Q_h as

$$\mathcal{G}_1 \begin{pmatrix} \vec{q}_n^{\,1} \\ \vec{q}_n^{\,2} \end{pmatrix} = \begin{pmatrix} \vec{F}_n^1 \\ \vec{F}_n^2 \end{pmatrix} - \sum_{k=1}^{n-1} \mathcal{G}_{n-k} \begin{pmatrix} \vec{q}_k^{\,1} \\ \vec{q}_k^{\,2} \end{pmatrix}, \qquad n \in \mathbb{Z}_+, \tag{6.29}$$

where the vectors F_n^1 and F_n^2 are defined by

$$(\vec{F}_n^{\kappa_1})_\alpha = \int_\Gamma \nu_\alpha(x) \int_{(n-1)h_t}^{nh_t} F(x,t) \rho_n^\kappa(t) dt dx, \qquad 1 \le \alpha \le N_x,\ \kappa_1 = 1, 2.$$

Since piecewise,*continuous* linears are discontinuous linears which satisfy

$$\vec{q}_k^{\,2} = \vec{q}_{k+1}^{\,1}, \qquad k \in \mathbb{Z}_+,$$

the form of the Galerkin equations in the basis of continuous linears may be deduced from (6.29). A more direct approach defines the hat functions

$$\rho_l^c(t) = \rho_{l+1}^1(t) + \rho_l^2(t), \qquad l \in \mathbb{N},$$

(where we set $\rho_0^2(t) = 0$), and explicitly computes the integrals. Because

$$\langle \rho_n^c, \mathcal{K} \rho_k^c \rangle = 0 \quad \text{for all} \quad n \ge k+1, \tag{6.30}$$

it is not hard to determine that the Galerkin equations would have a (partitioned) lower Hessenberg form. In other words, they would have non-zero matrices on the superdiagonal. These terms would seem to preclude any simple and economical solution scheme.

Remark: The Hessenberg form of Galerkin equations in this basis sharply contrasts with the lower triangular form of the matrix equations obtained using continuous linear polynomials and the method of point collocation in time. On the other hand, there is some reason to believe that degree of polynomials needed by a collocation method to achieve a given order of accuracy must be larger than the degree of polynomials used in a Galerkin method, (see [1]). Thus, the question of method selection appears to be more delicate than it would originally seem.

7 Error Analysis of the Galerkin Method

The error analysis of the Galerkin method follows directly from the quasioptimality (6.3) and appropriate approximation theory. For generality, we will consider the general trial spaces $Q_h^{d_x,d_t}$. We begin by recalling some standard facts from approximation theory.

7.1 Approximation Theory in the Anisotropic Sobolev Spaces

For the approximation error in $V_h^{d_t}$, we have [3], [4, Theorem 4.12],[6, Lemma 5.1], [1, p. 46]

$$\|u - P_t^{d_t} u\|_{H^{\beta_1}(\mathbb{R}_+)} \leq C h_t^{\beta_2 - \beta_1} \|u\|_{H^{\beta_2}(\mathbb{R}_+)}, \qquad u \in H^{\beta_2}(\mathbb{R}), \tag{7.1}$$

for all values of (β_1, β_2) which satisfy

$$-(d_t + 1) \leq \beta_1 < \beta_2 \leq d_t + 1, \quad \beta_2 > -1/2, \quad \text{and} \quad \beta_1 < 1/2. \tag{7.2}$$

The corresponding error estimates for the projection operator $P_x^{d_x}$, assuming sufficient smoothness on Γ, are (also see [28], [46])

$$\|u - P_x u\|_{H^{\alpha_1}(\Gamma)} \leq C h_x^{\alpha_2 - \alpha_1} \|u\|_{H^{\alpha_2}(\Gamma)}, \qquad u \in H^{\alpha_2}(\Gamma), \tag{7.3}$$

for all (α_1, α_2) which satisfy

$$-(d_x + 1) \leq \alpha_1 < \alpha_2 \leq d_x + 1, \quad \alpha_2 > -1/2, \quad \text{and} \qquad \alpha_1 < 1/2. \tag{7.4}$$

In the next lemma, we bound the difference $u - P_x P_t u$ in negative anisotropic norms, that is, in $H^{\lambda,\mu}(\Gamma, \mathbb{R})$ norms for which $\lambda, \mu \leq 0$.

Lemma 7.1 *Let (λ, μ, r, s) denote values which satisfy*

$$-d_x \leq \lambda \leq 0 \leq r \leq d_x + 1,$$

and

$$-d_t \leq \mu \leq 0 \leq s \leq d_t + 1. \tag{7.5}$$

Then, for all $u \in H^{r,s}(\Gamma, \mathbb{R}_+)$, there exists a positive constant C which depends on (λ, μ, r, s) such that

$$\|u - P_x P_t u\|_{H^{\lambda,\mu}(\Gamma,\mathbb{R}_+)} \leq C \left(h_x^{-\lambda} + h_t^{-\mu}\right) \left(h_x^r + h_t^s\right) \|u\|_{H^{r,s}(\Gamma,\mathbb{R}_+)}. \tag{7.6}$$

Proof: Fix λ, μ, r, and s. We bound $u - P_x P_t u$ in the $L^2(\Gamma \times \mathbb{R}_+)$ norm by applying the triangle inequality to

$$u - P_x P_t u = (u - P_x u) + P_x(u - P_t u).$$

We get

$$\begin{aligned}\|u - P_x P_t u\|_{L^2(\Gamma\times\mathbb{R}_+)} &\le \|u - P_x u\|_{L^2(\Gamma\times\mathbb{R}_+)} + \|P_x(u - P_t u)\|_{L^2(\Gamma\times\mathbb{R}_+)} \\ &\le C\left(h_x^r \|u\|_{L^2(\mathbb{R}_+, H^r(\Gamma))} + h_t^s \|u\|_{H^s(\mathbb{R}_+, L^2(\Gamma))}\right),\end{aligned}$$

by (7.1) and (7.3). Thus,

$$\|u - P_x P_t u\|_{L^2(\Gamma\times\mathbb{R}_+)} \le C(h_x^r + h_t^s)\|u\|_{H^{r,s}(\Gamma,\mathbb{R}_+)}. \tag{7.7}$$

To deduce bounds on $u - P_x P_t u$ in the $H^{\lambda,\mu}(\Gamma, \mathbb{R}_+)$ norm, we use duality. By definition, we have

$$\begin{aligned}\|u - P_x P_t u\|_{H^{\lambda,\mu}(\Gamma,\mathbb{R}_+)} &= \sup_{v\in H_0^{-\lambda,-\mu}(\Gamma,\mathbb{R}_+)} \frac{|\langle u - P_x P_t u, v\rangle|}{\|v\|_{H^{-\lambda,-\mu}(\Gamma,\mathbb{R}_+)}} \\ &= \sup_{v\in H_0^{-\lambda,-\mu}(\Gamma,\mathbb{R}_+)} \frac{|\langle u, v - P_x P_t v\rangle|}{\|v\|_{H^{-\lambda,-\mu}(\Gamma,\mathbb{R}_+)}}.\end{aligned} \tag{7.8}$$

(Recall that $H_0^{-\lambda,-\mu}(\Gamma, \mathbb{R}_+)$ is the completion of test functions $\mathcal{D}(\Gamma \times \mathbb{R}_+)$ in the $H^{-\lambda,-\mu}(\Gamma, \mathbb{R}_+)$ norm.) Substituting

$$\begin{aligned}|\langle u, v - P_x P_t v\rangle| &\le \|v - P_x P_t v\|_{L^2(\Gamma\times\mathbb{R}_+)} \|u\|_{L^2(\Gamma\times\mathbb{R}_+)} \\ &\le C(h_x^{-\lambda} + h_t^{-\mu})\|v\|_{H^{-\lambda,-\mu}(\Gamma,\mathbb{R}_+)} \|u\|_{L^2(\Gamma\times\mathbb{R}_+)}, \quad \text{by (7.7)},\end{aligned}$$

into (7.8), we get

$$\|u - P_x P_t u\|_{H^{\lambda,\mu}(\Gamma,\mathbb{R}_+)} \le C(h_x^{-\lambda} + h_t^{-\mu})\|u\|_{L^2(\Gamma\times\mathbb{R}_+)}. \tag{7.9}$$

Finally, to show (7.6), we combine (7.7-7.9) with the identity

$$(I - P_x P_t)^2 = (I - P_x P_t),$$

as follows:

$$\begin{aligned}\|u - P_x P_t u\|_{H^{\lambda,\mu}(\Gamma,\mathbb{R}_+)} &= \|(I - P_x P_t)^2 u\|_{H^{\lambda,\mu}(\Gamma,\mathbb{R}_+)} \\ &\le C(h_x^{-\lambda} + h_t^{-\mu})\|u - P_x P_t u\|_{L^2(\Gamma,\mathbb{R}_+)}, \quad \text{by (7.9)}, \\ &\le C(h_x^{-\lambda} + h_t^{-\mu})(h_x^r + h_t^s)\|u\|_{H^{r,s}(\Gamma,\mathbb{R}_+)}, \quad \text{by (7.7)}. \ \square\end{aligned}$$

In the next two lemmas, we consider $u - P_x P_t u$ in $H^{\lambda,\mu}(\Gamma, \mathbb{R})$ norms for positive values of λ and μ.

Lemma 7.2 *Let (λ, μ, r, s) denote values which satisfy*

$$0 \le \lambda \le r \le d_x + 1, \qquad \lambda < d_x + 1/2, \tag{7.10}$$

$$0 \le \mu \le s \le d_t + 1, \qquad \mu < d_t + 1/2. \tag{7.11}$$

Then, for all

$$u \in H^{\lambda,\mu}(\Gamma, \mathbb{R}_+) \cap H^{\mu}(\mathbb{R}, H^{r-\lambda}(\Gamma)) \cap H^{s-\mu}(\mathbb{R}_+, H^{\lambda}(\Gamma)), \tag{7.12}$$

there exists a positive constant C such that

$$\begin{aligned} \|u - P_x P_t u\|_{H^{\lambda,\mu}(\Gamma,\mathbb{R}_+)} \le\ & C(h_x^{r-\lambda} + h_t^{s-\mu})(\|u\|_{H^{r,s}(\Gamma,\mathbb{R}_+)} \\ & + \|u\|_{H^{\mu}(\mathbb{R}_+,H^{s-\lambda}(\Gamma))} \\ & + \|u\|_{H^{r-\mu}(\mathbb{R}_+,H^{\lambda}(\Gamma))}) \end{aligned} \tag{7.13}$$

Proof: Fix the values of λ, μ, r and s and set

$$\alpha = r - \lambda \quad \text{and} \quad \beta = s - \mu.$$

Assuming that u satisfies (7.12), we first apply the triangle inequality to

$$u - P_x P_t u = (u - P_x u) + P_x(u - P_t u),$$

to get

$$\begin{aligned} \|u - P_x P_t u\|_{H^{\mu}(\mathbb{R}_+,L^2(\Gamma))} &\le \|u - P_x u\|_{H^{\mu}(\mathbb{R}_+,L^2(\Gamma))} \\ &\quad + \|P_x(u - P_t u)\|_{H^{\mu}(\mathbb{R}_+,L^2(\Gamma))} \\ &\le C(h_x^{\alpha}\|u\|_{H^{\mu}(\mathbb{R}_+,H^{\alpha}(\Gamma))} \\ &\quad + h_t^{\beta}\|u\|_{H^{s}(\mathbb{R}_+,L^2(\Gamma))}). \end{aligned} \tag{7.14}$$

Analogously, applying the triangle inequality to

$$u - P_x P_t u = (u - P_t u) + P_t(u - P_x u),$$

we get

$$\begin{aligned} \|u - P_x P_t u\|_{L^2(\mathbb{R}_+,H^{\lambda}(\Gamma))} \le\ & C(h_t^{\beta}\|u\|_{H^{\beta}(\mathbb{R}_+,H^{\lambda}(\Gamma))} \\ & + h_x^{\alpha}\|u\|_{L^2(\mathbb{R}_+,H^{r}(\Gamma))}). \end{aligned} \tag{7.15}$$

Clearly, (7.13) follows from (7.15) and (7.14). □

Though (7.13) is optimal with respect to powers of h_x and h_t, it is somewhat unsatisfactory since it requires more than $u \in H^{r,s}(\Gamma, \mathbb{R}_+)$ regularity. Actually, we can use interpolation theory to show the next result.

Lemma 7.3 *Let (λ, μ, r, s) satisfy (7.11) and*

$$\frac{r}{s} = \frac{\lambda}{\mu}. \tag{7.16}$$

Then, there exist positive constants $C_1(r,s)$ and $C_2(r,s)$ such that

$$\|u\|_{H^{\mu}(\mathbb{R}_+, H^{r-\lambda})(\Gamma)} \leq C_1(r,s)\|u\|_{H^{r,s}(\Gamma,\mathbb{R}_+)} \tag{7.17}$$

and

$$\|u\|_{H^{s-\mu}(\mathbb{R}_+, H^{\lambda}(\Gamma))} \leq C_2(r,s)\|u\|_{H^{r,s}(\Gamma,\mathbb{R}_+)}. \tag{7.18}$$

Hence, for such values of (λ, μ, r, s), the approximation error (7.13) satisfies the symmetrical form

$$\|u - P_x P_t u\|_{H^{\lambda,\mu}(\Gamma,\mathbb{R}_+)} \leq C(r,s)(h_x^{r-\lambda} + h_t^{s-\mu})\|u\|_{H^{r,s}(\Gamma,\mathbb{R}_+)}.$$

Proof: Let $u \in H^{r,s}(\Gamma, \mathbb{R}_+)$. This means that

$$u \in H^0(\mathbb{R}_+, H^r(\Gamma)) \qquad \text{and} \quad u \in H^s(\mathbb{R}_+, H^0(\Gamma)).$$

By interpolation theory, (see [24, chapter 1] and [25, chapter 4]), it follows that

$$u \in H^{\sigma s}(\mathbb{R}_+, H^{(1-\sigma)r}(\Gamma)) \quad \text{for each} \quad \sigma \in [0,1].$$

Hence, by setting $\sigma = \mu/s$ and then $\sigma = 1 - \mu/s$, we get

$$\|u\|_{H^{\mu}(\mathbb{R}_+, H^{r-\lambda}(\Gamma))} \leq C_1\|u\|_{H^{r,s}(\Gamma,\mathbb{R}_+)},$$

and

$$\|u\|_{H^{s-\mu}(\mathbb{R}_+, H^{\lambda}(\Gamma))} \leq C_2\|u\|_{H^{r,s}(\Gamma,\mathbb{R}_+)}. \ \square$$

Besides approximation estimates, we need to develop an inverse inequality for $Q_h^{d_x,d_t}$. For such an inequality, we must henceforth assume that the triangulation of Γ is quasi-uniform. Informally, this means that the ratio of the maximum to minimum diameter of the partition is bounded from above and below. Practically, this is not a restrictive assumption on the triangulation and implies the inverse inequalities [1, pp. 359-360], [9], [46],

$$\|u\|_{H^{\alpha_2}(\Gamma)} \leq C h_x^{-(\alpha_2-\alpha_1)} \|u\|_{H^{\alpha_1}(\Gamma)} \quad \text{for all} \quad u \in X_h^{d_x}, \tag{7.19}$$

for all values $-\infty \leq \alpha_1 \leq \alpha_2 < 1/2$. We also have

$$\|u\|_{H^{\beta_2}(\mathbb{R})} \leq C h_t^{-(\beta_2-\beta_1)} \|u\|_{H^{\beta_1}(\mathbb{R})} \quad \text{for all} \quad u \in V_h^{d_t}, \tag{7.20}$$

for all values $-\infty \leq \beta_1 \leq \beta_2 < 1/2$. From these results, we deduce the next lemma.

Lemma 7.4 *Let $\lambda, \mu \geq 0$. Then, there exists a positive constant $C(\lambda, \mu)$ which is independent of the subspace $Q_h^{d_x,d_t}$ such that*

$$\|q\|_{L^2(\Gamma\times\mathbb{R})} \leq C(\lambda, \mu) \max\left(h_x^{-\lambda}, h_t^{-\mu}\right) \|q\|_{H^{-\lambda,-\mu}}, \qquad q \in Q_h^{d_x,d_t}. \tag{7.21}$$

Proof: Let $q \in Q_h^{d_x,d_t}$ be given. We use (7.19) and (7.20) in

$$\|q\|^2_{H^{\lambda,\mu}(\Gamma,\mathbb{R})} = \|q\|^2_{H^{\mu}(\mathbb{R},L^2(\Gamma))} + \|q\|^2_{L^2(\mathbb{R},H^{\lambda}(\Gamma)}, \qquad \lambda,\, \mu > 0,$$

to get

$$\begin{aligned} \|q\|^2_{H^{\lambda,\mu}(\Gamma,\mathbb{R})} &\leq C(\lambda,\mu)(h_x^{-2\lambda} + h_t^{-2\mu})\|q\|^2_{L^2(\Gamma\times\mathbb{R})} \\ &\leq C(\lambda,\mu)(h_x^{-\lambda} + h_t^{-\mu})^2\|q\|^2_{L^2(\Gamma\times\mathbb{R})}. \end{aligned} \tag{7.22}$$

Substituting (7.22) on the right of the inequality

$$\|q\|^2_{L^2(\Gamma,\mathbb{R})} \leq \|q\|_{H^{\lambda,\mu}(\Gamma,\mathbb{R})}\|q\|_{H^{-\lambda,-\mu}(\Gamma,\mathbb{R})}, \qquad \lambda, \mu \geq 0, \tag{7.23}$$

we get

$$\begin{aligned} \|q\|_{L^2(\Gamma\times\mathbb{R})} &\leq C(\lambda,\mu)(h_x^{-\lambda} + h_t^{-\mu})\|q\|_{H^{-\lambda,-\mu}(\Gamma,\mathbb{R})} \\ &\leq C(\lambda,\mu)\max(h_x^{-\lambda}, h_t^{-\mu})\|q\|_{H^{-\lambda,-\mu}(\Gamma,\mathbb{R})}. \ \Box \end{aligned}$$

7.2 Error Estimates I

The purpose of this section is to give error estimates in a variety of anisotropic norms between the Galerkin approximation $q_h \in Q_h^{d_x,d_t}$ and the exact solution to $\mathcal{K}_1 q = F$. Our first result which estimates $q - q_h$ in the so called energy norm $H^{-1/2,-1/4}(\Gamma, \mathbb{R}_+)$ clearly follows from (6.3) and (7.6).

Theorem 7.5 *Let $q_h \in Q_h^{d_x,d_t}$ denote the Galerkin approximation to $\mathcal{K}_1 q = F$. Then,*

$$\|q - q_h\|_{H^{-1/2,-1/4}(\Gamma,\mathbb{R}_+)} \le C\|q - P_x P_t q_h\|_{H^{-1/2,-1/4}(\Gamma,\mathbb{R}_+)}.$$

Thus, if $q \in H^{d_x+1,d_t+1}(\Gamma, \mathbb{R}_+)$,

$$\|q - q_h\|_{H^{-1/2,-1/4}(\Gamma,\mathbb{R}_+)} \le C(h_x^{1/2} + h_t^{1/4})(h_x^{d_x+1} + h_t^{d_t+1})\|q\|_{H^{d_x+1,d_t+1}(\Gamma,\mathbb{R}_+)}. \tag{7.24}$$

We remark that if $d_x + 1 = 2(d_t + 1)$ then (7.24) becomes

$$\|q - q_h\|_{H^{-1/2,-1/4}(\Gamma,\mathbb{R}_+)} \le C(h_x^2 + h_t)^{d_t+5/4}\|q\|_{H^{2(d_t+1),d_t+1}(\Gamma,\mathbb{R}_+)}.$$

Now, in the main theorem of this section, we consider the error $q - q_h$ in the $L^2(\Gamma \times \mathbb{R})$ norm.

Theorem 7.6 *Assume that $Q_h^{d_x,d_t}$ satisfies the inverse inequality (7.21) and that $q \in H^{d_x+1,d_t+1}(\Gamma, \mathbb{R}_+)$. Then, the Galerkin solution q_h satisfies*

$$\|q - q_h\|_{L^2(\Gamma\times\mathbb{R}_+)} \le \mathcal{C}(h_x, h_t)(h_x^{d_x+1} + h_t^{d_t+1})\|q\|_{H^{d_x+1,d_t+1}(\Gamma,\mathbb{R}_+)}. \tag{7.25}$$

where

$$\mathcal{C}(h_x, h_t) \le C \max\left[\left(\frac{h_x^2}{h_t}\right)^{1/4}, \left(\frac{h_x^2}{h_t}\right)^{-1/4}\right]. \tag{7.26}$$

Proof: For clarity, we will number the constants which appear, using the letters C to denote constants which are independent of the stepsizes h_x, h_t and the letters $\mathcal{C}$ to denote specific constants which depend on (h_x, h_t).

To estimate the difference $q - q_h$, we write

$$q - q_h = q - P_x P_t q + P_x P_t q - q_h.$$

Applying the triangle inequality, we get

$$\|q - q_h\|_{L^2(\Gamma\times\mathbb{R}_+)} \le \|q - P_xP_tq\|_{L^2(\Gamma\times\mathbb{R}_+)} + \|P_xP_tq - q_h\|_{L^2(\Gamma\times\mathbb{R}_+)}.$$

By (7.7), we have

$$\|q - P_xP_tq\|_{L^2(\Gamma\times\mathbb{R}_+)} \le C_1(h_x^{d_x+1} + h_t^{d_t+1})\|q\|_{H^{d_x+1,d_t+1}(\Gamma,\mathbb{R}_+)}. \tag{7.27}$$

Thus, it remains to estimate $q_h - P_xP_tq \in Q_h$. By the inverse inequality (7.21), we have

$$\|q_h - P_xP_tq\|_{L^2(\Gamma\times\mathbb{R}_+)} \le C_2 \max(h_x^{-1/2}, h_t^{-1/4})\|q_h - P_xP_tq\|_{H^{-1/2,-1/4}(\Gamma,\mathbb{R}_+)}. \tag{7.28}$$

Since $q_h - P_xP_tq = (q - P_xP_tq) - (q - q_h)$, equations (7.6) and (7.24) show that

$$\begin{aligned}\|q_h - P_xP_tq\|_{H^{-1/2,-1/4}(\Gamma,\mathbb{R}_+)} &\le \|q - P_xP_tq\|_{H^{-1/2,-1/4}(\Gamma,\mathbb{R}_+)} \\ &\quad + \|q - q_h\|_{H^{-1/2,-1/4}(\Gamma,\mathbb{R}_+)} \\ &\le C_3(h_x^2 + h_t)^{1/4}(h_x^{d_x+1} + h_t^{d_t+1})\|q\|_{H^{d_x+1,d_t+1}}.\end{aligned}$$

We now substitute this inequality into (7.28). This yields

$$\|P_xP_tq - q_h\|_{L^2(\Gamma\times\mathbb{R}_+)} \le \mathcal{C}_1(h_x, h_t)(h_x^{d_x+1} + h_t^{d_t+1})\|q\|_{H^{d_x+1,d_t+1}(\Gamma,\mathbb{R}_+)}. \tag{7.29}$$

where

$$\mathcal{C}_1(h_x, h_t) \le C_2C_3 \max(h_x^{-1/2}, h_t^{-1/4})(h_x^{1/2} + h_t^{1/4}).$$

Thus, by combining (7.27) with (7.29), we conclude that

$$\|q - q_h\|_{L^2(\Gamma\times\mathbb{R}_+)} \le \mathcal{C}_2(h_x, h_t)(h_x^{d_x+1} + h_t^{d_t+1})\|q\|_{H^{d_x+1,d_t+1}(\Gamma,\mathbb{R}_+)}, \tag{7.30}$$

where

$$\mathcal{C}_2(h_x, h_t) = \mathcal{C}_1(h_x, h_t) + C_1.$$

To conclude the proof, we note that

$$\begin{aligned}\max(h_x^{-1/2}, h_t^{-1/4})(h_x^{1/2} + h_t^{1/4}) &\le 2\max(h_x^{-1/2}, h_t^{-1/4})\max(h_x^{1/2}, h_t^{1/4}) \\ &= 2\max\left[\left(\frac{h_x^2}{h_t}\right)^{1/4}, \left(\frac{h_x^2}{h_t}\right)^{-1/4}\right],\end{aligned}$$

for all h_x, $h_t > 0$. Hence, it follows that

$$\mathcal{C}_2(h_x, h_t) \leq C_4 \max\left[\left(\frac{h_x^2}{h_t}\right)^{1/4}, \left(\frac{h_x^2}{h_t}\right)^{-1/4}\right], \tag{7.31}$$

for some positive constant C_4. Substituting (7.31) into (7.30) proves the theorem. □

The factor

$$\max\left[\left(\frac{h_x^2}{h_t}\right)^{1/4}, \left(\frac{h_x^2}{h_t}\right)^{-1/4}\right],$$

seems to suggest that the ratio h_x^2/h_t must remain bounded throughout computations in order to achieve optimal order convergence in $L^2(\Gamma \times \mathbb{R}_+)$.

7.3 The Aubin-Nitsche Lemma and Interior Error Estimates

In this section, we apply the Aubin-Nitsche lemma to derive error estimates between the Galerkin solution q_h and the exact solution q in lower Sobolev spaces. We then use these estimates to deduce L^∞ error estimates between

$$u(x,t) = \int_0^t \int_\Gamma K(x-y, t-s) q(y,s) dy ds, \qquad x \in \mathbb{R}^3,\ t > 0, \tag{7.32}$$

and

$$u_h(x,t) = \int_0^t \int_\Gamma K(x-y, t-s) q_h(y,s) dy ds, \qquad x \in \mathbb{R}^3,\ t > 0. \tag{7.33}$$

Theorem 7.7 *Let $q_h \in Q_h^{d_x,d_t}$ denote the Galerkin approximation to $\mathcal{K}_1 q = F$. Then, for all $0 \leq \mu \leq \min(d_t + 5/4, d_x/2 + 3/4)$, there exists a positive constant C such that*

$$\|q - q_h\|_{H^{-(1/2+2\mu),-(1/4+\mu)}(\Gamma,\mathbb{R}_+)} \leq C(h_x^2 + h_t)^\mu \|q - q_h\|_{H^{-1/2,-1/4}(\Gamma,\mathbb{R}_+)}. \tag{7.34}$$

Thus, if $q \in H^{d_x+1,d_t+1}(\Gamma, \mathbb{R}_+)$,

$$\|q - q_h\|_{H^{-(1/2+2\mu),-(1/4+\mu)}(\Gamma,\mathbb{R}_+)} \leq \mathcal{C}(\mu, d_x, d_t, h_t, h_x)\|q\|_{H^{d_x+1,d_t+1}(\Gamma,\mathbb{R}_+)},$$

where

$$\mathcal{C}(\mu, d_x, d_t, h_t, h_x) \leq C(h_x^2 + h_t)^\mu (h_t^{1/4} + h_x^{1/2})(h_x^{d_x} + h_t^{d_t}). \tag{7.35}$$

Proof: For brevity, we introduce some temporary notation. For all real numbers β, we let

$$X^{\beta} = H^{2\beta,\beta}(\Gamma, \mathbb{R}_+) \quad \text{and} \quad \tilde{X}^{\beta} = H^{2\beta,\beta}(\Gamma, \mathbb{R}).$$

We further define the spaces

$$X_0^{\beta} = H_0^{2\beta,\beta}(\Gamma, \mathbb{R}_+), \qquad \beta \geq 0.$$

Recall that $H_0^{2\beta,\beta}(\Gamma, \mathbb{R}_+)$ was defined as the completion of $\mathcal{D}(\Gamma \times \mathbb{R}_+)$ in the $H^{2\mu,\mu}(\Gamma, \mathbb{R}_+)$ norm. Therefore,

$$X^{-\beta} = \left(X_0^{\beta}\right)^*, \qquad \beta \geq 0.$$

Set $\mu_0 = \min(d_t + 5/4, d_x/2 + 3/4)$. For any $0 \leq \mu \leq \mu_0$, we have

$$\|q - q_h\|_{X^{-(1/4+\mu)}} = \sup_{p \in X_0^{1/4+\mu}} \frac{|\langle q - q_h, p\rangle|}{\|p\|_{X_0^{1/4+\mu}}}.$$

Let R_+: $\tilde{X}^{\beta} \to X^{\beta}$ denote the restriction operator and E_+: $X^{\beta} \to \tilde{X}^{\beta}$ any (fixed) operator of extension. Since $R_+E_+p = p$ almost everywhere, we have

$$\begin{aligned} \|q - q_h\|_{X^{-(1/4+\mu)}} &= \sup_{p \in X_0^{1/4+\mu}} \frac{|\langle q - q_h, R_+E_+p\rangle|}{\|p\|_{X_0^{1/4+\mu}}} \\ &= \sup_{p \in X_0^{1/4+\mu}} \frac{|\langle R_+^*(q - q_h), E_+p\rangle|}{\|p\|_{X_0^{1/4+\mu}}}. \end{aligned}$$

Since E_+ is bounded, it follows that

$$\|q - q_h\|_{X^{-(1/4+\mu)}} \leq C \sup_{p \in X_0^{1/4+\mu}} \frac{|\langle R_+^*(q - q_h), E_+p\rangle|}{\|E_+p\|_{\tilde{X}^{1/4+\mu}}}.$$

Setting $\rho = E_+p$, it therefore follows that

$$\|q - q_h\|_{X^{-(1/4+\mu)}} \leq C \sup_{\rho \in \tilde{X}^{1/4+\mu}} \frac{|\langle R_+^*(q - q_h), \rho\rangle|}{\|\rho\|_{\tilde{X}^{1/4+\mu}}}. \qquad (7.36)$$

Now, define $\theta \in \tilde{X}^{\mu-1/4}$ as the unique solution to the adjoint equation $\tilde{\mathcal{K}}_1^*\theta = \rho$. Since $\tilde{\mathcal{K}}_1^*$: $\tilde{X}^{\mu-1/4} \to \tilde{X}^{\mu+1/4}$ is an isomorphism, there exists some positive constant c such that

$$c\|\rho\|_{\tilde{X}^{\mu+1/4}} \leq \|\theta\|_{\tilde{X}^{\mu-1/4}}.$$

Therefore, substituting this into (7.36), we have

$$\begin{aligned}\|q - q_h\|_{X^{-(1/4+\mu)}} &\leq C \sup_{\theta\in\tilde{X}^{\mu-1/4}} \frac{|\langle R_+^*(q-q_h), \tilde{\mathcal{K}}_1^*\theta\rangle|}{\|\theta\|_{\tilde{X}^{\mu-1/4}}} \\ &= C \sup_{\theta\in\tilde{X}^{\mu-1/4}} \frac{|\langle \tilde{\mathcal{K}}_1 R_+^*(q-q_h), \theta\rangle|}{\|\theta\|_{\tilde{X}^{\mu-1/4}}}. \qquad (7.37)\end{aligned}$$

We now use the fact that the Galerkin solution q_h satisfies

$$\langle \tilde{\mathcal{K}}_1 R_+^*(q-q_h), \theta\rangle = 0 \quad \text{for all} \quad \theta \in \tilde{Q}_h.$$

(This is easily seen since R_+^* corresponds to, or more precisely extends, the zero extension operator.) Therefore,

$$\begin{aligned}\|q - q_h\|_{X^{-(1/4+\mu)}} &\leq C \sup_{\theta\in\tilde{X}^{\mu-1/4}} \frac{|\langle \tilde{\mathcal{K}}_1 R_+(q-q_h), (I - P_x P_t)\theta\rangle|}{\|\theta\|_{\tilde{X}^{\mu-1/4}}} \\ &\leq C\|\tilde{\mathcal{K}}_1 R_+^*(q-q_h)\|_{\tilde{X}^{1/4}} \frac{\|\theta - P_x P_t\theta\|_{\tilde{X}^{-1/4}}}{\|\theta\|_{\tilde{X}^{\mu-1/4}}}. \quad (7.38)\end{aligned}$$

The boundedness of the operators $\tilde{\mathcal{K}}_1$ and R_+^* imply

$$\|\tilde{\mathcal{K}}_1 R_+^*(q-q_h)\|_{X^{1/4}} \leq C\|q - q_h\|_{H^{-1/2,-1/4}(\Gamma,\mathbb{R}_+)}.$$

Using this estimate and the approximation inequality

$$\|\theta - P_x P_t\theta\|_{H^{-1/2,-1/4}(\Gamma,\mathbb{R}_+)} \leq C(h_x^2 + h_t)^\mu \|\theta\|_{\tilde{X}^{\mu-1/4}}, \qquad 0 \leq \mu \leq \mu_0, \quad (7.39)$$

in (7.38), we get

$$\|q - q_h\|_{H^{-(1/2+2\mu),-(1/4+\mu)}(\Gamma,\mathbb{R}_+)} \leq C(h_x^2 + h_t)^\mu \|q - q_h\|_{H^{-1/2,-1/4}(\Gamma,\mathbb{R}_+)}.$$

This shows (7.34). Assuming the greater regularity on q, we get the next result by applying the energy estimate (7.24). □

We make two brief remarks. First, observe that it is the approximation inequality (7.39) which limits the range of μ. Secondly, we can only simplify the constant $\mathcal{C}$ in special cases. For example, if $d_x + 1 = 2(d_t + 1)$, it can be written in the more attractive form

$$\mathcal{C}(\mu, d_x, d_t, h_t, h_x) \leq C(h_x^2 + h_t)^{\mu+5/4+d_t}, \qquad 0 \leq \mu \leq d_t + 5/4.$$

Theorem 7.8 *Let $q \in H^{-1/2,-1/4}(\Gamma, \mathbb{R}_+)$ denote the solution to $\mathcal{K}_1 q = F$ and $q_h \in Q_h^{d_x,d_t}$ the Galerkin solution. Define u and u_h by (7.32) and (7.33). Then, for all $x \in \mathbb{R}^3$ such that dist$(\Gamma, x) \geq \delta > 0$ and for each $0 \leq \mu \leq \min(d_t + 5/4, d_x/2 + 3/4)$, there exists a positive constant $C(\delta, \mu)$ such that*

$$|u(x,t) - u_h(x,t)| \leq C(\delta, \mu)\mathcal{C}(\mu, d_x, d_t, h_x, h_t)\|q\|_{H^{d_x,d_t}(\Gamma,\mathbb{R}_+)}, \tag{7.40}$$

where

$$\mathcal{C}(\mu, d_x, d_t, h_t, h_x) \leq C(h_x^2 + h_t)^{\mu}(h_t^{1/4} + h_x^{1/2})(h_x^{d_x} + h_t^{d_t}).$$

Proof: Set $\mu_0 = \min(d_t + 5/4, d_x/2 + 3/4)$. We subtract (7.33) from (7.32) and take absolute values. We get

$$\begin{aligned} |u(x,t) - u_h(x,t)| &\leq |\int_0^t \int_\Gamma K(x-y, t-s)(q - q_h)(y,s)dyds| \\ &\leq C\|K\|_{X^{\mu+1/4}}\|q - q_h\|_{X^{-(1/4+\mu)}} \end{aligned}$$

for any $0 \leq \mu \leq \mu_0$. Now, because x is bounded away from Γ, the heat kernel K is a C^∞ function which decays sufficiently rapidly at infinity. Hence, it belongs to X^s for any $s \in \mathbb{R}$. The proof now follows by applying estimate (7.34) to the difference $q - q_h$. □

7.4 Error Estimates II

The analysis of the previous section assumed that no approximation of the forcing function F (in the equation $\mathcal{K}_1 q = F$) occurred. This is not a reasonable assumption, even if one neglects the effects of numerical integration. Often, a better model is to assume that the Galerkin solution q_h solves the equations

$$\langle p, \mathcal{K}_1 q_h\rangle = \langle p, F_h\rangle \quad \text{for all} \quad p \in Q_h \subset H^{-1/2,-1/4}(\Gamma, \mathbb{R}_+), \tag{7.41}$$

where F_h is some approximation of F. In the next lemma, we adjust the error bounds between q_h and $q = \mathcal{K}_1^{-1}F$ to reflect this additional approximation. Since the proof is obvious, we omit the details.

Lemma 7.9 *Let $F_h \in H^{1/2,1/4}(\Gamma, \mathbb{R}_+)$ and let q_h denote the solution to the Galerkin equations (7.41). Then,*

$$\begin{aligned} \|q - q_h\|_{H^{-1/2,-1/4}(\Gamma,\mathbb{R}_+)} &\le C\{\|q - P_x P_t q_h\|_{H^{-1/2,-1/4}(\Gamma,\mathbb{R}_+)} \\ &\qquad + \|F - F_h\|_{H^{1/2,1/4}(\Gamma,\mathbb{R}_+)}\}. \end{aligned} \tag{7.42}$$

Similarly, for a quasi-uniform mesh,

$$\begin{aligned} \|q - q_h\|_{L^2(\Gamma,\mathbb{R}_+)} &\le \mathcal{C}\{\|q - P_x P_t q_h\|_{L^2(\Gamma,\mathbb{R}_+)} \\ &\qquad + \|F - F_h\|_{H^{1/2,1/4}(\Gamma,\mathbb{R}_+)}\}, \end{aligned} \tag{7.43}$$

where $\mathcal{C}$ is defined in (7.26).

In the rest of this section, we examine the error term $F - F_h$ for our application of the Galerkin method to the direct integral equation. That is, we study the difference between $F = (\frac{1}{2} + \mathcal{K}_2)g + \mathcal{M}f$ and $F_h = (\frac{1}{2} + \mathcal{K}_2)P_x P_t g + \mathcal{M}_{\mathcal{T}}\Pi_1 f$. (Recall that $\Pi_1 f$ denotes the linear interpolant of f with respect to a isoparametric triangulation $\mathcal{T}$ of Ω and $P_x P_t g$ a tensor product interpolation of g by piecewise constants in time and piecewise linears in space.)

We consider the error due to our approximation of the domain term first. We begin with a preliminary lemma.

Lemma 7.10 *Let O denote any bounded, open set such that*

$$O \subset \{x \in \mathbb{R}^3 \colon |x| \le R\}, \qquad \text{some } R > 0.$$

For any $f \in L^2(O)$, let

$$U(x,t) = \int_O K(x - x', t) f(x') dx', \qquad x \in \mathbb{R}^3,\ t > 0, \tag{7.44}$$

where K denotes the fundamental solution to the heat equation. Then, there exists a positive constant $C(R)$ such that

$$\|U\|^2_{V(\mathbb{R}^3,\mathbb{R}_+)} \le C(R)\|f\|^2_{L^2(O)}. \tag{7.45}$$

Proof: Let

$$\tilde{f}(x) = \begin{cases} f(x), & x \in O, \\ 0, & x \in \mathbb{R}^3 \backslash O, \end{cases}$$

so that

$$U(x,t) = \int_{\mathbb{R}^3} K(x - x', t)\tilde{f}(x')dx', \qquad x \in \mathbb{R}^3,\ t > 0.$$

We take the Fourier transform in space of both sides. This yields

$$\hat{U}_x(\xi, t) = e^{-|\xi|^2 t}\mathcal{F}_x\left(\tilde{f}\right)(\xi), \qquad \xi \in \mathbb{R}^3,\ t > 0. \tag{7.46}$$

where $\hat{U}_x$ denotes the Fourier transform in space of U.

Since

$$\int_0^\infty e^{-2|\xi|^2 t}dt = \frac{1}{2|\xi|},$$

it follows that

$$\int_0^\infty \int_{\mathbb{R}^3} (1 + |\xi|^2)|\hat{U}_x(\xi, t)|^2 d\xi dt = \frac{1}{2}\int_{\mathbb{R}^3} \frac{(1 + |\xi|^2)}{|\xi|^2}|\mathcal{F}_x\left(\tilde{f}\right)(\xi)|^2 d\xi, \tag{7.47}$$

To bound the right hand side of (7.47), note that $\tilde{f}$ having compact support implies

$$\begin{aligned} |\mathcal{F}_x\left(\tilde{f}\right)(\xi)| &\leq |\int_{\mathbb{R}^3} \tilde{f}(x)e^{-ix\xi}dx| \\ &\leq \int_O |f(x)|dx \\ &\leq C(R)\int_O |f(x)|^2 dx. \end{aligned}$$

Since (B_1 denotes the unit ball in $\mathbb{R}^3$)

$$\begin{aligned} \int_{\mathbb{R}^3} \frac{(1 + |\xi|^2)}{|\xi|^2}|\mathcal{F}_x\left(\tilde{f}\right)(\xi)|^2 d\xi &= \int_{B_1} \frac{(1 + |\xi|^2)}{|\xi|^2}|\mathcal{F}_x\left(\tilde{f}\right)(\xi)|^2 d\xi \\ &\quad + \int_{\mathbb{R}^3\backslash B_1} \frac{(1 + |\xi|^2)}{|\xi|^2}|\mathcal{F}_x\left(\tilde{f}\right)(\xi)|^2 d\xi \\ &\leq C\|f\|^2_{L^2(O)}\int_{B_1}(|\xi|^{-2} + 1)d\xi \\ &\quad + 2\int_{\mathbb{R}^3\backslash B_1} |\mathcal{F}_x\left(\tilde{f}\right)(\xi)|^2 d\xi, \end{aligned}$$

it follows that

$$\int_{\mathbb{R}^3} \frac{(1+|\xi|^2)}{|\xi|^2} |\mathcal{F}_x \left(\tilde{f}\right)(\xi)|^2 \leq C\|f\|^2_{L^2(O)}.$$

Applying this in (7.47), we have

$$\|U\|^2_{L^2(\mathbb{R}^3,\mathbb{R}_+)} \leq C(R)\|f\|^2_{L^2(O)}, \qquad (7.48)$$

for some positive constant $C(R)$. Analogously, because

$$\mathcal{F}_x\left(\frac{\partial U}{\partial t}\right)(\xi,t) = -2|\xi|^2 e^{-|\xi|^2 t}\mathcal{F}_x\left(\tilde{f}\right)(\xi), \qquad \xi \in \mathbb{R}^3,\ t>0, \qquad (7.49)$$

we have

$$\begin{aligned} \|\frac{\partial U}{\partial t}\|^2_{L^2(\mathbb{R}_+,H^{-1}(\mathbb{R}^3))} &= \int_0^\infty \int_{\mathbb{R}^3} (1+|\xi|^2)^{-1} |\mathcal{F}_x\left(\frac{\partial U}{\partial t}\right)(\xi,t)|^2 d\xi dt \\ &\leq \int_0^\infty \int_{\mathbb{R}^3} \frac{|\xi|^4}{|\xi|^2+1} e^{-2|\xi|^2 t} |\mathcal{F}_x\left(\tilde{f}\right)(\xi)|^2 d\xi dt \\ &\leq \frac{1}{2}\|\tilde{f}\|^2_{L^2(\mathbb{R}^3)} \\ &= \frac{1}{2}\|f\|^2_{L^2(O)}. \ \square \qquad (7.50) \end{aligned}$$

Using Lemma 7.10, we can now analyze the error between $\mathcal{M}f$ and $\mathcal{M}_\mathcal{T}\Pi_1 f$.

Theorem 7.11 *Let Ω denote an open, bounded set in $\mathbb{R}^3$ and $f \in L^\infty(\Omega)$. Set*

$$E(x,t) = \mathcal{M}f(x,t) - \mathcal{M}_\mathcal{T}\Pi_1 f(x,t), \qquad x \in \mathbb{R}^3,\ t>0.$$

Then, there exists a positive constant C which depends on Ω such that

$$\|E\|_{V(\mathbb{R}^3,\mathbb{R}_+)} \leq C(\text{meas}(\Omega - \Omega_\mathcal{T}) + h^2_\mathcal{T})\|f\|_{H^2(\mathbb{R}^3)}. \qquad (7.51)$$

Proof: For each $(x,t) \in \mathbb{R}^3 \times \mathbb{R}_+$, let

$$E_1(x,t) \ = \ \mathcal{M}f(x,t) - \mathcal{M}_\mathcal{T} f(x,t),$$

and

$$E_2(x,t) = \mathcal{M}_{\mathcal{T}} f(x,t) - \mathcal{M}_{\mathcal{T}} \Pi_1 f(x,t).$$

Clearly, $E = E_1 + E_2$. By Lemma 7.10, we have

$$\begin{aligned} \|E_2\|_{V(\mathbb{R}^3,\mathbb{R}_+)} &\leq C(\Omega)\|f - \Pi_1 f\|_{L^2(\mathbb{R}^3,\mathbb{R}_+)} \\ &\leq C(\Omega) h_{\mathcal{T}}^2 \|f\|_{H^2(\Omega)}, \end{aligned} \tag{7.52}$$

by familiar approximation theory [9, Section 4.2].

Thus, it remains to estimate E_1. Set

$$\bar{f}(x) = \begin{cases} f(x), & x \in \Omega \backslash \Omega_{\mathcal{T}}, \\ 0, & \text{otherwise}, \end{cases}$$

for each $x \in \mathbb{R}^3$. Then, we have

$$\begin{aligned} E_1(x,t) &= \int_{\Omega\backslash\Omega_{\mathcal{T}}} K(x-x',t) f(x') dx', \\ &= \int_{\Omega} K(x-x',t) \bar{f}(x') dx', \end{aligned}$$

for all $x \in \mathbb{R}^3$, $t > 0$. Hence, by Lemma 7.10,

$$\|E_2\|_{V(\mathbb{R}^3,\mathbb{R}_+)}^2 \leq C \|\bar{f}\|_{L^2(\Omega)}^2, \tag{7.53}$$

for some positive constant C. The theorem now follows since

$$\begin{aligned} \|\bar{f}\|_{L^2(\Omega)}^2 &= \int_{\Omega\backslash\Omega_{\mathcal{T}}} |f(x')|^2 dx' \\ &\leq [\ \text{meas}(\Omega - \Omega_{\mathcal{T}})\ ]^2\ \|f\|_{L^\infty(\mathbb{R}^3)}^2, \end{aligned}$$

and

$$\|f\|_{L^\infty(\mathbb{R}^3)} \leq C \|f\|_{H^2(\mathbb{R}^3)},$$

by a Sobolev embedding theorem [16, p. 243]. □

Since $I + \mathcal{K}_2$: $H^{1/2,1/4}(\Gamma,\mathbb{R}_+) \to H^{1/2,1/4}(\Gamma,\mathbb{R}_+)$ is bounded, the error due to our approximation of the double layer term is easily deduced. We have

$$\begin{aligned} \|(1/2 + \mathcal{K}_2)(g - P_x P_t g)\|_{H^{1/2,1/4}(\Gamma,\mathbb{R}_+)} &\leq C\|g - P_x P_t g\|_{H^{1/2,1/4}(\Gamma,\mathbb{R}_+)} \\ &\leq C(h_x^2 + \tau)^{3/4} \|g\|_{H^{2,1}(\Gamma,\mathbb{R}_+)} \end{aligned}$$

8 Numerical Examples

In this section, we report the results of some numerical experiments. Since these experiments are preliminary in nature, we shall consider the analogous single layer potential $\mathcal{K}_1$ which is defined on surfaces $\Gamma \in \mathbb{R}^2$. (The operator defined by (1.11) with the fundamental solution now given by

$$K(x,t) = \begin{cases} \dfrac{\exp(-|x|^2/4t)}{(4\pi t)} & x \in \mathbb{R}^2,\, t > 0 \\ 0 & x \in \mathbb{R}^2,\, t < 0 \end{cases} .)$$

The objective of these experiments is to solve the initial-Dirichlet value boundary value heat equations

$$\begin{aligned} \frac{\partial u}{\partial t}(x,t) - \Delta u(x,t) &= 0, && x \in \Omega,\, t > 0, \\ u(x,0) &= 0, && x \in \Gamma, \\ u(x,t) &= g(x,t), && x \in \Gamma,\, t > 0, \end{aligned} \tag{8.1}$$

using the direct integral equation

$$\mathcal{K}_1 q(x,t) = \frac{1}{2} g(x,t) + \mathcal{K}_2 g(x,t), \qquad x \in \Gamma \subset \mathbb{R}^2,\, t > 0.$$

Throughout, our basic assumption on Γ is that there exists a 1-periodic parametrization $\Phi(\theta)$: $\theta \in [0,1] \to \Gamma$. To recall, our method consists in replacing the given Dirichlet data by its projection g_h in the subspace Q_h of piecewise linear in space and piecewise constant in time and to then solve the Galerkin equations

$$\langle v, \mathcal{K}_1 q_h \rangle = \langle v, \frac{1}{2} g_h(x,t) + \mathcal{K}_2 g_h(x,t) \rangle \quad \text{for all} \quad v \in Q_h,$$

for $q_h \in Q_h$. As shown in the text, this procedure reduces to solving the set of linear systems

$$\sum_{k=1}^{n} G_{n,k} \vec{q}_k = \sum_{k=0}^{n} L_{n,k} \vec{g}_n, \qquad \text{for each } n \in \mathbb{Z}_+, \tag{8.2}$$

where $\vec{q}_n$ and $\vec{g}_n$ represent average values of the approximate Neumann data q_h and given Dirichlet data g at time step n.

To accurately determine the convergence of the method, the examples chosen have known solutions. In general, except for the first run, each run consists of two parts. In the first part, we solve (8.2) for the fluxes q_n. In the second part, we then use the representation formula

$$\begin{aligned} u(x,t) \; &= \; \int_0^t \int_\Gamma \left[\frac{\partial u}{\partial \mathbf{n}}(y,t')K(x-y,t-t') \mp \frac{\partial K}{\partial \mathbf{n}}(x-y,t-t')u(y,t') \right] dydt', \\ &\qquad x \in \mathbb{R}^3 \backslash \Gamma, t > 0, \end{aligned}$$

to recover approximations to the corresponding solutions u. (The minus sign holds for the interior problem, while the plus sign holds for the exterior problem.)

In the first three of our examples, Ω is the unit disk. We will describe the points of the disk by polar coordinates (r, θ), with the angular variable θ scaled to lie between 0 and 1. In this case, exact solutions to (8.1) are available in series form. We supply some details on the construction of these solutions in appendix C. In each of these examples, approximations were sought over the space-time cylinder $\partial B_1 \times (0, 4)$.

Example 1:

Dirichlet data

$$g(\theta, t) = t^2.$$

Exact solution

$$\begin{aligned} u(r,\theta,t) \;\; &= \;\; t^2 - 4 \sum_{k=1}^{\infty} \frac{J_0(\alpha_k r)}{\alpha_k^3 J_1(\alpha_k)} \left[t - \frac{1}{\alpha_k^2}(1 - e^{-\alpha_k^2 t}) \right], \\ &\text{where} \qquad J_0(\alpha_k) = 0. \end{aligned}$$

Exact Boundary Flux

$$q(\theta, t) = t - 4 \sum_{k=0}^{\infty} \frac{(1 - e^{-\alpha_k^2 t})}{\alpha_k^4}. \tag{8.3}$$

(Note: $\partial J_0(z)/\partial z = -J_1(z)$.)

This example was ran for debugging purposes since it could be compared to runs previously reported in the literature [42], [34]. Note that the Dirichlet

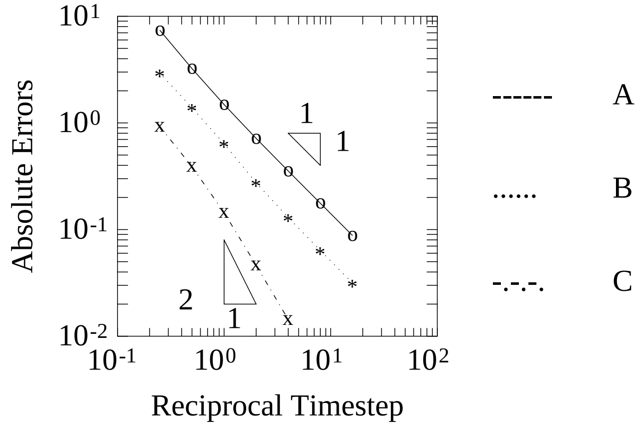

Figure 1: Absolute Errors versus Reciprocal Timestep $1/h_t$. A) L_2 Error. B) L^∞ Error. C) Midpoint l^∞ Error.

data g and the exact flux q are independent of the θ coordinate. This is reflected in numerical computations since each component of the vectors q_n were found to be identical. Considering that the Galerkin method makes no spatial discretization error, a reasonable assumption is that the number of boundary nodes can remain at constant value independent of the timestep h_t. We found this observation to be true with 8 boundary nodes sufficient.

In Figure 1, we graphically examine the rates of convergence between the Galerkin solution and the exact flux in both the L^2 and L^∞ norms. Clearly, the rates of convergence are linear in both norms. Since the L^∞ comparison between the Galerkin and exact solution is pessimistic, we also looked at the maximun of the absolute errors $|q(t_{n-1/2}) - q_h(t_{n-1/2})|$ at the midpoints $t_{n-1/2} = (n - 1/2)h_t$. Here, we see evidence that the rate of convergence at the midpoints is faster. To determine the rate, we looked at the midpoint errors at selective points. In Table 8, we display the relative errors at the points $t = 1.$ and $t = 3.0$. These results show quadratic convergence with respect to the time step.

h_t	*Rel Error*	*Rate*
2	4.44601e−01	
2/3	2.82556e−02	2.5085
2/9	4.52148e−03	1.6679
2/27	5.06173e−04	1.9932
2/81	5.66493e−05	1.9934

h_t	*Rel Error*	*Rate*
2	5.6458e−02	
2/3	8.7235e−03	1.699
2/9	9.7534e−04	1.994
2/27	1.1071e−04	1.980
2/81	1.2486e−05	1.986

Table 0.1: Relative L^∞ Errors as a Function of h_t

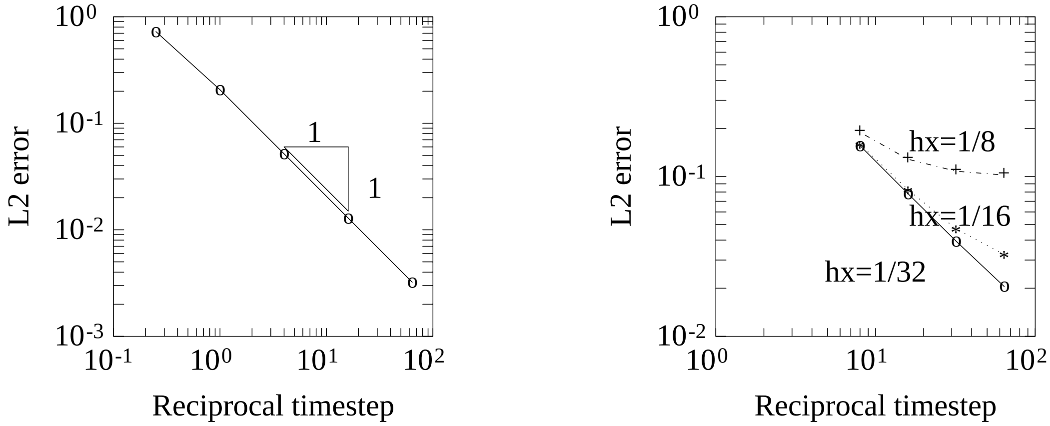

Figure 2: Relative L^2 Errors versus Reciprocal Timestep

Example 2:

Dirichlet data

$$g(\theta,t) = t^2 \cos 2\pi\theta.$$

Exact solution

$$u(x,t) \;=\; \left\{ rt^2 - 4\sum_{k=1}^{\infty} \frac{J_1(\beta_k r)}{\beta_k^3 J_1(\beta_k)} \left[t - \frac{1}{\beta_k^2}(1 - e^{-\beta_k^2 t}) \right] \right\} \cos 2\pi\theta,$$

$$\text{where} \qquad J_1(\beta_k) = 0.$$

Exact Boundary Flux

$$q(\theta,t) = \left\{ t^2 - .25t + 4\sum_{k=1}^{\infty} \frac{(1 - e^{-\beta_k^2 t})}{\beta_k^4} \right\} \cos 2\pi\theta.$$

We start by investigating the total L^2 approximation error over the cylinder $\partial B_1 \times (0,4)$. Since the optimal rate of convergence to expect is $O(h_x^2 + h_t)$, we performed a sequence of runs in which the time and spatial steps were related by $8h_x = h_t^{1/2}$. The results are shown in the left graph of Figure 2. They indicate that the method is converging linearly with respect to h_t.

Considering the excessive demands on storage this algorithm makes, (each successive run in Figure 2 increased the demand on storage by a factor of 16), it is important to recognize the relative importance of the stepsizes h_t and h_x. We performed a set of experiments with various values of h_t and h_x. In order to conserve a bit of storage, we reduced the time axis of interest in this investigation to $(0, 1/2)$. The results are graphically summarized in

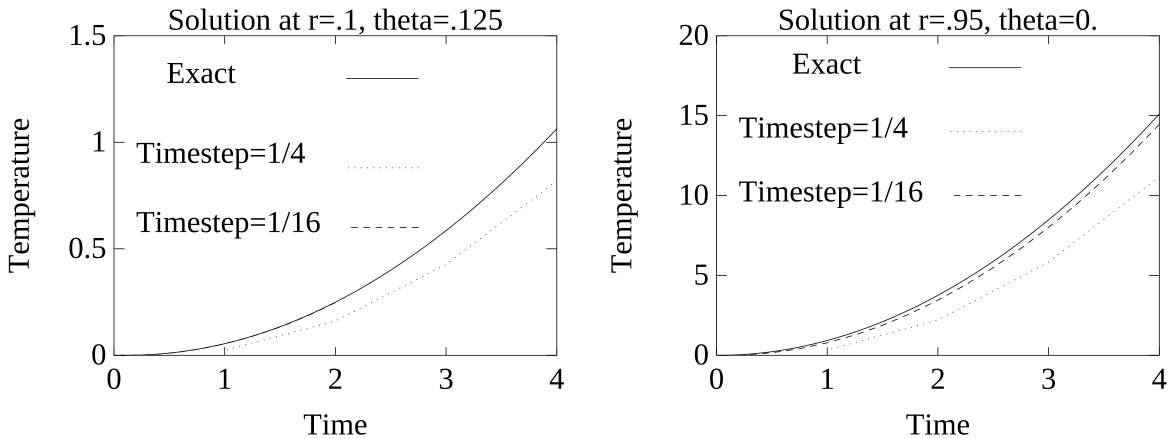

Figure 3: Interior Approximations

Figure 2. They indicate that a consistent choice of time and space step is needed for efficiency.

We now consider the approximation of the solutions in the interior. To recall, our approximation consists in collocating the representation formula at a desired interior point (x, t) (or points) and then replacing the densities g and q by their piecewise and Galerkin interpolants. Since the point x is away from the boundary, the spatial integrals can be computed using a order 4 Gaussian rule.

We determine the solution as a function of time for various values of r and θ. Figure 3 clearly shows how the rate of convergence is slower for the point closest to the boundary Γ. Observe how the Galerkin approximation with $h_t = 1/16$ is identical to the exact solution when $r = .1$.

Example 3:

Dirichlet data

$$g(\theta, t) = \cos 2\pi\theta$$

Exact solution

$$u(r, \theta, t) = r \cos 2\pi\theta \tag{8.4}$$

Exact Boundary Flux

$$q(\theta, t) = \cos 2\pi\theta \tag{8.5}$$

This example illustrates the effect a discontinuity between the boundary and initial data has on the solutions. In Figure 4, we show the Galerkin approximations to the flux and to the interior solution.

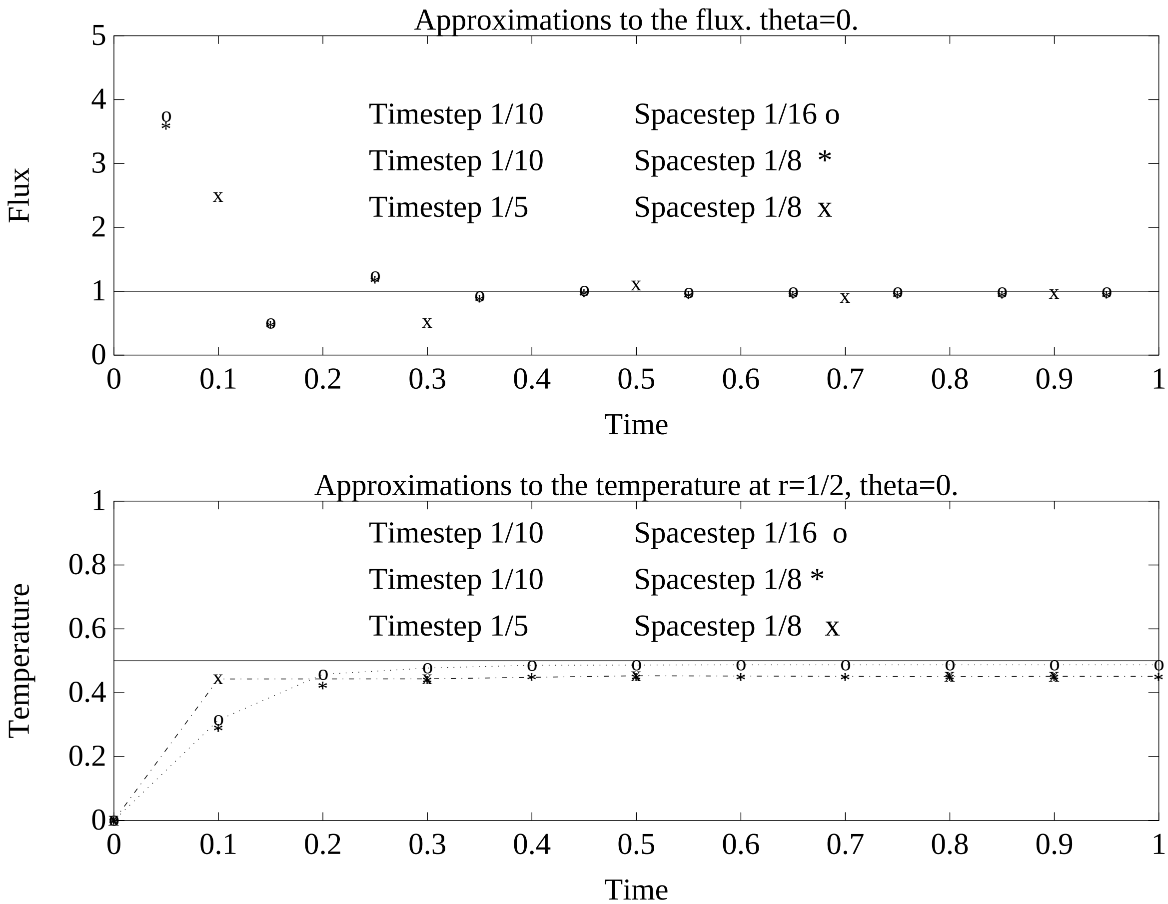

Figure 4: Galerkin Approximations

Clearly, the method behaves quite poorly near the origin $t = 0$. Fortunately, however, the effect of this discontinuity decays fairly rapidly with time. Indeed, away from the origin, we found the method to be converging optimally.

Example 4: In this example, we consider an exterior problem. It is for such exterior problems that the boundary element method seems to be most useful. We take the boundary Γ to be the ellipse

$$E_0 := \left\{ \quad (x,y) : \frac{x^2}{x_0^2} + \frac{y^2}{y_0^2} = 1 \quad \right\},$$

with $(x_0, y_0) = (3/2, 1)$.

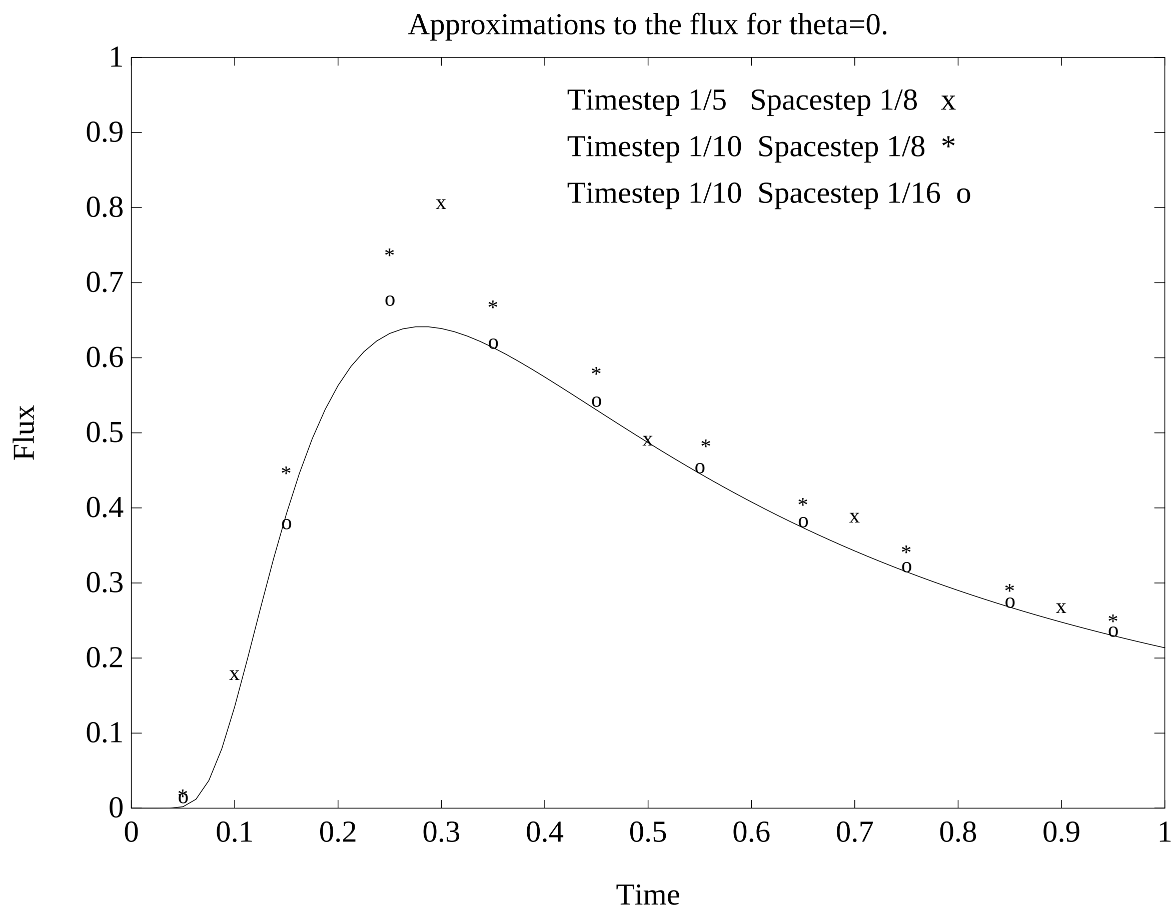

Figure 5: Galerkin Approximation to Flux

In order to have a problem with a known solution, we choose the Dirichlet g so that the exact solution u to the heat equation in the exterior is the fundamental solution

$$u(x,t) = \frac{\exp(-|x|^2/4t)}{2t}, \qquad x \in \mathbb{R}^2,\ t > 0.$$

Note that g has both a spatial and time dependence since the radius $\rho(\theta)$ of E_0 is

$$\rho(\theta) = \left\{ x_0^2 \cos^2 2\pi\theta + y_0^2 \sin^2 2\pi\theta \right\}^{1/2}. \tag{8.6}$$

It is easy to check that the exact Neumann flux is

$$q(\theta,t) = \frac{\exp(-\rho^2(\theta)/4t)}{4t^2\left[x_0^2\sin^2(2\pi\phi) + y_0^2\cos^2(2\pi\phi)\right]^{1/2}}.$$

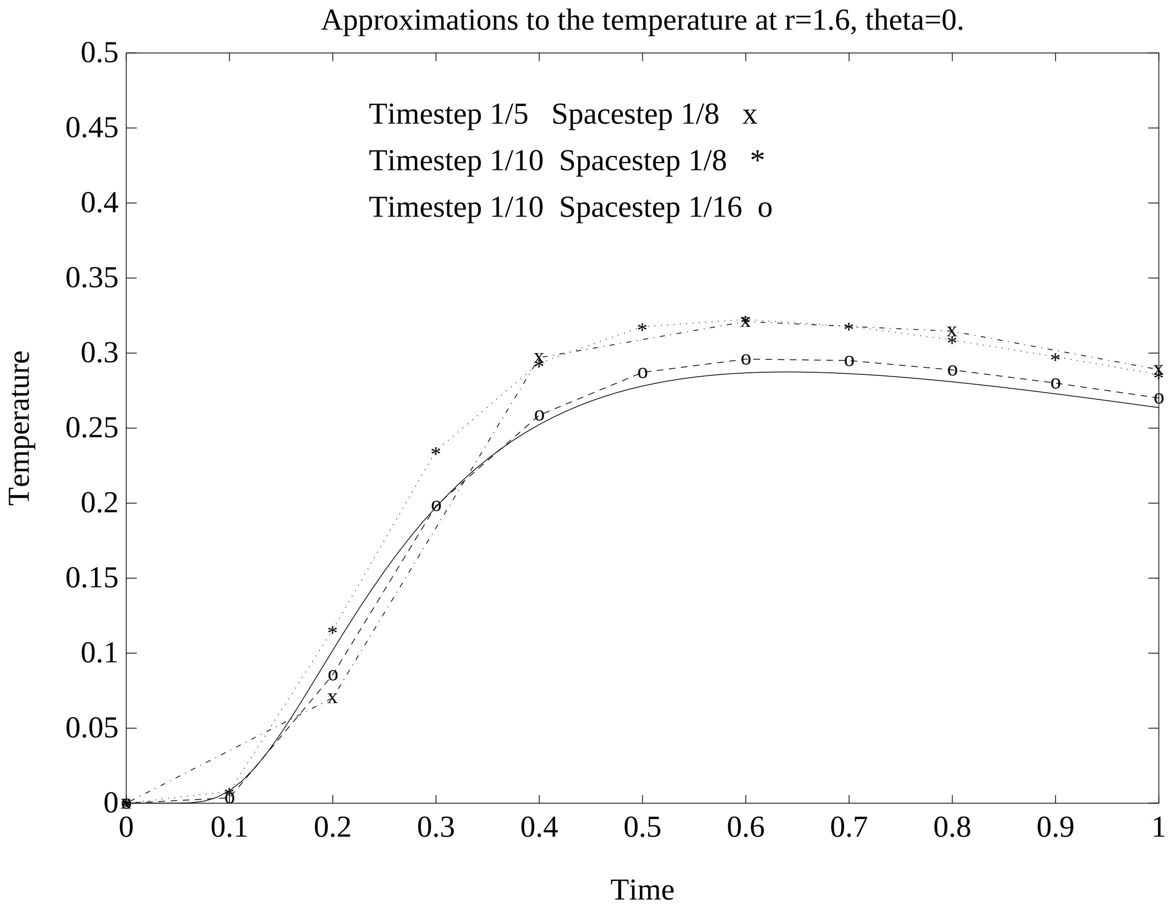

Figure 6: Galerkin Approximation to Solutions

In Figure 5, we consider the Galerkin approximations to the flux. The results indicate that its spatial step which is controlling the error. The same remarks can be made about the approximations of the solution. Some of these are shown in Figure 6.

We conclude with some observations. In the first example, we saw evidence that the Galerkin method when applied to smooth functions not only converges optimally in L^2 by also exhibits superconvergence at the midpoints. In the second example, we saw the competitive effect of the discretizations in space and time. In the third example, we examined how the Galerkin method behaves when applied to a problem with incompatible Dirichlet and initial data. Very encouragingly, we saw that although it behaves quite poorly near the discontinuity, the Galerkin method damps out its effects fairly rapidly. Lastly, we applied the method to an exterior problem to show how the boundary element is well suited to such problems.

A Proof of Theorem 5.3

In this appendix, we give the proof of Theorem 5.3 in Section 5.1. Since the proof employs the method of finite differences, we begin with a quick review of these operators before proving the theorem. For each integer $i = 1, 2, 3$, and any $h > 0$, the first order difference operator $d_{y_i}(h)$ is defined on $H^{1,1/2}(\mathbb{R}^3, \mathbb{R})$ by

$$d_{y_i}(h)w(y,t) = \frac{w(y + h\mathbf{e}_i, t) - w(y,t)}{h}, \qquad y \in \mathbb{R}^3,\ t \in \mathbb{R},$$

where $\mathbf{e}_i$ denotes the unit vector in the y_i direction.

The higher order difference operators are defined by successive applications of this map. For any multiindex β and any set H of positive real numbers $(h_{ij}, i = 1, 2, 3,\ j = 1, \ldots, \beta_i)$ the notation

$$D_H^\beta w = \prod_{i=1}^{3} \prod_{j=1}^{\beta_i} d_{y_i}(h_{ij})w, \qquad w \in H^{1,1/2}(\mathbb{R}^3, \mathbb{R}),$$

defines the finite difference operators D_H^β. Set

$$|\beta| = \sum_{j=1}^{3} |\beta_j|,$$

and let $|H|$ denote the largest number in the set H. The following lemma relates D_H^β to the partial differential operator

$$\partial^\beta = \frac{\partial^{\beta_1}}{\partial y_1} \frac{\partial^{\beta_2}}{\partial y_2} \frac{\partial^{\beta_3}}{\partial y_3}.$$

For more details, see [16, pg 258].

Lemma A.1 *Let r, s be any non-negative real numbers and suppose that $w \in H^{r,s}(\mathbb{R}^3, \mathbb{R})$ satisfies*

$$\limsup_{|H| \to 0} \|D_H^\beta w\|_{H^{r,s}} \le C_0,$$

for some finite constant C_0 and some multiindex β. Then, $\partial^\beta w \in H^{r,s}(\mathbb{R}^3, \mathbb{R})$ and satisfies

$$\|\partial^\beta w\|_{H^{r,s}(\mathbb{R}^3, \mathbb{R})} \le C_0.$$

We note some properties satisfied by the difference operators. If γ denotes the restriction operator, (i.e.,

$$\gamma w(y_1, y_2, t) = w(y_1, y_2, 0, t), \qquad (y, t) \in \mathbb{R}^3 \times \mathbb{R},)$$

and $\beta = (\beta_1, \beta_2, \beta_3)$ any multiindex such that $\beta_3 = 0$, then

$$\gamma D_H^\beta w(y_1, y_2, y_3, t) = D_H^\beta \gamma w(y_1, y_2, t).$$

Another useful property of the finite difference operator is that it commutes with differentiation, i.e.

$$D_H^\beta \partial^\alpha w = \partial^\alpha D_H^\beta w, \quad \text{for any multiindices} \quad \alpha,\ \beta,$$

for $w \in \mathcal{S}(\mathbb{R}^3 \times \mathbb{R})$, say. Finally, by a simple change of variables, it is easy to prove the important equality

$$\begin{aligned} \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} D_H^\beta w(y,t) v(y,t) dy dt &= (-1)^{|\beta|} \int_{\mathbb{R}^3} w(y,t) D_{-H}^\beta v(y,t) dy dt, \\ &\text{for all} \quad w, v \in L^2(\mathbb{R}^3 \times \mathbb{R}). \end{aligned} \tag{A.1}$$

We are now ready to begin the proof. First, we remind the reader of some previous notation:

$$\begin{aligned} \Lambda &= \text{Heat operator } \mathcal{W}^{1,1/2} \to \mathcal{W}^{-1,-1/2}, \\ \gamma &= \text{Trace operator } \mathcal{W} \to H^{1/2,1/4}(\Gamma \times \mathbb{R}), \\ Y &= \{y \in \mathbb{R}^3 \colon |y_i| \le 1, i = 1, 2, 3\}, \\ Y_0 &= \{y \in Y \colon y_3 = 0\}, \\ O &= \text{An open set in } \mathbb{R}^3. \text{ A representative} \\ &\quad\ \text{of the sets in a finite covering of } \Omega, \\ \phi &= \text{Isomorphism from } O \to Y\ . \end{aligned}$$

Here is what we want to prove. Given $m \in \mathbb{N}$ and $q \in H^{m-1/2, m-1/4}(\Gamma, \mathbb{R})$, let

$$w_q(y,t) = \phi^*(\zeta u_q)(y,t), \qquad y \in Y,\ t \in \mathbb{R}.$$

Theorem 5.3 states that all of the partial derivatives $\partial^\beta w_q$ exist and belong to $L^2(\mathbb{R}, H^1(Y))$ so long as $|\beta| \le m$ and $\beta_3 = 0$. Moreover, they each satisfy an inequality of the form

$$\|\partial^\beta w_q\|_{H^{1,1/2}(Y,\mathbb{R})} \le C \|q\|_{H^{m-1/2, m/2-1/4}(\Gamma,\mathbb{R})}. \tag{A.2}$$

Proof of Theorem 5.3: Let $H_0^{1,1/2}(Y,\mathbb{R})$ denote the closure of $\mathcal{D}(Y\times\mathbb{R})$ in the $H^{1,1/2}(Y,\mathbb{R})$ norm. (In other words, functions with compact support in $Y\times\mathbb{R}$.) Define $\mathcal{A}$: $H_0^{1,1/2}(Y,\mathbb{R})\times H_{0,}^{1,1/2}(Y,\mathbb{R})\to\mathbb{C}$ by

$$\mathcal{A}(w,v)=\langle\Lambda\psi^*(w),\psi^*(v)\rangle\,. \tag{A.3}$$

It is simple to check that $\mathcal{A}$ can be written in the form

$$\begin{aligned}\mathcal{A}(w,v) &= \int_{-\infty}^{\infty}\int_Y i\tau\mathcal{F}_t(w)(y,\tau)\overline{\mathcal{F}_t(v)(y,\tau)}J(y)dyd\tau \\ &\quad +\sum_{i,j}^{3}\int_{-\infty}^{\infty}\int_Y a_{ij}(y)\frac{\partial w}{\partial y_i}(y,t)\overline{\frac{\partial v}{\partial y_j}}(y,t)J(y)dydt,\end{aligned} \tag{A.4}$$

for each $w,v\in H^{1,1/2}(Y,\mathbb{R})$, where $J(y)$ denotes the Jacobian of ψ and where $a_{i,j}(y)\in C^\infty(Y)$ with the matrix of coefficients $\{a_{i,j}(y)\}$ being uniformly positive definite. (That is, there exists a positive constant c which is independent of $y\in Y$ such that

$$\sum_{i,j}^{3}\nu_i a_{i,j}(y)\nu_j\geq c\sum_{i=1}^{3}\nu_i^2\quad\text{for all}\quad\nu\in\mathbb{R}^3.) \tag{A.5}$$

Consequently, there exists some positive constant c such that

$$\mathcal{A}(w,(I-\mathcal{H})w)\geq c\|w\|^2_{H^{1,1/2}(Y,\mathbb{R})}\quad\text{for all}\quad w\in H_{0,}^{1,1/2}(Y,\mathbb{R}), \tag{A.6}$$

where $\mathcal{H}$: $H_0^{1,1/2}(Y,\mathbb{R})\to H_0^{1,1/2}(Y,\mathbb{R})$ denotes the Hilbert transform.

We substitute w_q for w in (A.3). Using the product rule, we get

$$\begin{aligned}\mathcal{A}(w_q,v) &= \langle\Lambda u_q,\zeta\psi^*(v)\rangle+\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}u_q(x,t)\nabla\zeta(x)\overline{\nabla\psi^*(v)\,(x,t)}dxdt \\ &\quad -\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\overline{\psi^*\,(v)\,(x,t)}\nabla\zeta(x)\nabla u_q(x,t)dxdt,\end{aligned}$$

where $u_q=\Lambda^{-1}\gamma^*q$. Hence,

$$\begin{aligned}\mathcal{A}(w_q,v) &= \langle\gamma^*q,\zeta\psi^*\,(v)\rangle+\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}u_q(x,t)\nabla\zeta(x)\overline{\nabla\psi^*\,(v)\,(x,t)}dxdt \\ &\quad -\int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\overline{\psi^*\,(v)\,(x,t)}\nabla\zeta(x)\nabla u_q(x,t)dxdt,\end{aligned} \tag{A.7}$$

for all $v \in H_0^{1,1/2}(Y,\mathbb{R})$.

To estimate $D_H^\beta w_q$, we will consider the pairing $\mathcal{A}(D_H^\beta w_q, v)$. Before we can take this step, however, we must verify that this pairing makes sense for all multiindices β such that $|\beta| \leq m$. Fortunately, this only requires a careful choice of the set H so that $|H|$ is sufficiently small. To be precise, let $r < 1$ be such that

$$\text{supp } w_q(\cdot, t) \subset Y_r := \{y \in \mathbb{R}^3 \colon |y_i| \leq r, i = 1,2,3\} \subset Y,$$

uniformly with respect to $t \in \mathbb{R}$. (The definition of w_q shows that such an r exists.) Set

$$\sigma_1 = \frac{1}{3} + \frac{2}{3}r, \quad \text{and} \quad \sigma_2 = \frac{2}{3} + \frac{1}{3}r,$$

so that $r < \sigma_1 < \sigma_2 < 1$. Then, it is easy to check that $D_H^\beta w_q$ has support in the set Y_{σ_2} for all sets H which satisfy

$$|H| \leq \frac{1}{3m}(1-r).$$

Moreover, we have

$$D_{-H}^\beta D_H^\beta w_q \in H_0^{1,1/2}(Y_{\sigma_2}, \mathbb{R}),$$

which we will need later.

Let $v \in H_0^{1,1/2}(Y,\mathbb{R})$ be given. Without loss of generality, we choose v so that its support lies in the cylinder $Y_{\sigma_2} \times \mathbb{R}$. (This ensures the inclusion

$$D_H^\beta v \subset H_0^{1,1/2}(Y,\mathbb{R}) \quad \text{for any} \quad \text{multiindex } \beta.)$$

We now consider the pairing $\mathcal{A}(D_H^\beta w_q, v)$. By (A.4) and (A.1), we have

$$\mathcal{A}(D_H^\beta w_q, v) = (-1)^{|\beta|}\mathcal{A}(w_q, D_{-H}^\beta v) + E_1(w_q, v), \tag{A.8}$$

where

$$\begin{aligned} E_1(w_q, v) \; &= \; \prod_{l=1}^{3}\prod_{k=1}^{\beta_l}\sum_{i,j}^{3}\int_{-\infty}^{\infty}\int_Y \frac{\partial w_q}{\partial y_i}(y,t)\overline{\frac{\partial v}{\partial y_j}}(y-h_{lk},t)D^l_{-h_{lk}}a_{i,j}(y)dydt \\ &+ \; \prod_{l=1}^{3}\prod_{k=1}^{\beta_j}\int_{-\infty}^{\infty}\int_Y i\tau\mathcal{F}_t(w_q)(y,\tau)\overline{\mathcal{F}_t(v)(y-h_{lk},\tau)}D^l_{-h_{lk}}\left(J(y)\right)dyd\tau. \end{aligned}$$

Using (A.7), we obtain

$$\mathcal{A}(D_H^\beta w_q, v) = \left\langle \gamma^* q, \zeta\psi^* \left(D_{-H}^\beta v\right)\right\rangle + E_1(w_q, v) + E_2(u_q, \psi^*(v)), \qquad (A.9)$$

where

$$\begin{aligned} E_2(u_q, \psi^*(v)) &= \int_{-\infty}^{\infty}\int_{\mathbb{R}^3} u_q(x,t)\nabla\zeta(x)\overline{\nabla v(\phi(x),t)}dxdt \\ &\quad - \int_{-\infty}^{\infty}\int_{\mathbb{R}^3} \overline{v(\phi(x),t)}\nabla\zeta(x)\nabla u_q(x,t)dxdt. \end{aligned}$$

We now bound the terms E_1, and E_2. We have

$$|E_1(w_q, v)| \le C\|w_q\|_{H^{1,1/2}(Y,\mathbb{R})}\|v\|_{H^{1,1/2}(Y,\mathbb{R})}, \qquad (A.10)$$

and

$$\begin{aligned} |E_2(u_q, v)| &\le C\left\{\|(\Delta\zeta)u_q\|_{L^2(\mathbb{R}^3\times\mathbb{R})} + \|\,|\nabla u_q|\,\|_{L^2(\mathbb{R}^3\times\mathbb{R})}\right\}\|v\|_{H^{1,0}(Y,\mathbb{R})} \\ &\le C\|u_q\|_{\mathcal{W}^{1,1/2}}\|v\|_{H^{1,1/2}(Y,\mathbb{R})}. \end{aligned} \qquad (A.11)$$

(To get the last estimate, note that $\Delta\zeta \in \mathcal{D}(\mathbb{R}^3)$ and apply Lemma 3.2.) Collecting together equations (A.8–A.11), it follows that

$$\mathcal{A}(D_H^\beta w_q, v) = \left\langle q, \gamma\zeta\psi^* \left(D_{-H}^\beta v\right)\right\rangle + \mathcal{E}(v), \quad v \in H^{1,1/2}(Y_{\sigma_2}, \mathbb{R}), \qquad (A.12)$$

where $\mathcal{E}$ satisfies

$$\begin{aligned} |\mathcal{E}(v)| &\le C\left\{\|u_q\|_{\mathcal{W}^{1,1/2}} + \|w_q\|_{H^{1,1/2}(Y,\mathbb{R})}\right\}\|v\|_{H^{1,1/2}(Y,\mathbb{R})} \\ &\le C\|q\|_{H^{-1/2,-1/4}(\Gamma,\mathbb{R})}\|v\|_{H^{1,1/2}(Y,\mathbb{R})}. \end{aligned} \qquad (A.13)$$

To proceed further, we now rewrite the term

$$I_q := \left\langle \gamma^* q, \zeta\psi^* \left(D_{-H}^\beta v\right)\right\rangle \qquad (A.14)$$

in a form where D_{-H}^β acts not on v but instead on $\phi^*(\zeta q)$. Set

$$V_{-H}^\beta(y', y_3, t) = D_{-H}^\beta(v)(y', y_3, t), \qquad (y', y_3, t) \in Y \times \mathbb{R}.$$

Then, from (A.14), we have

$$I_q = \int_{-\infty}^{\infty}\int_{\Gamma\cap O} q(x,t)\zeta(x)\overline{V_{-H}^\beta(\phi(x),t)}dxdt.$$

Changing the variable of integration to $y \in Y_0$ (by the mapping ψ), we get

$$I_q = \int_{-\infty}^{\infty} \int_{Y_0} \phi^* (\zeta q) (y', t) \overline{V^{\beta}_{-H}(y', 0, t)} J(y') dy' dt, \tag{A.15}$$

where we have written $V^{\beta}_{-H}(y', 0, t)$ for $\gamma \circ V^{\beta}_{-H}$.

Now, since $\beta_3 = 0$, the trace operator γ commutes with the finite difference operator, i.e.,

$$\begin{aligned} V^{\beta}_{-H}(y', 0, t) &= \gamma \circ D^{\beta}_{-H} (v) \\ &= D^{\beta}_{-H} \circ \gamma (v) \\ &= D^{\beta}_{-H} (\gamma v) (y', t). \end{aligned} \tag{A.16}$$

Therefore,

$$I_q = \int_{-\infty}^{\infty} \int_{Y_0} \phi^* (\zeta q) (y', t) D^{\beta}_{-H} (\gamma \overline{v}) (y', t) J(y') dy' dt. \tag{A.17}$$

Using (A.1) in this equation, we obtain

$$\begin{aligned} I_q \;=\; & \int_{-\infty}^{\infty} \int_{Y_0} D^{\beta}_{H} (\phi^* (\zeta q)) (y', t) \gamma \overline{v}(y', t) J(y') dy' dt \\ & + \prod_{j=1}^{2} \prod_{k=1}^{\beta_j} \int_{-\infty}^{\infty} \int_{Y_0} (\phi^* (\zeta q)) (y' + h_{jk}, t) \overline{\gamma v(y', t)} D^{j}_{-h_{jk}} J(y') dy' d\tau. \end{aligned}$$

Taking absolute values, we get

$$\begin{aligned} |I_q| &\le C \|q\|_{H^{m-1/2, m/2-1/4}(\Gamma, \mathbb{R})} \|\gamma v\|_{L^2(Y_0 \times \mathbb{R})} \\ &\le C \|q\|_{H^{m-1/2, m/2-1/4}(\Gamma, \mathbb{R})} \|v\|_{H^{1,1/2}(Y, \mathbb{R})}. \end{aligned} \tag{A.18}$$

Thus, (A.18) and (A.12) show the existence of some positive constant C such that

$$\begin{aligned} |\mathcal{A}(D^{\beta}_{H} w_q, v)| &\le C \|q\|_{H^{m-1/2, m/2-1/4}(\Gamma, \mathbb{R})} \|v\|_{H^{1,1/2}(Y, \mathbb{R})}, \\ & \qquad \text{for any} \quad v \in H^{1,1/2}_{0}(Y_{\sigma_2}, \mathbb{R}). \end{aligned} \tag{A.19}$$

But, by selecting $v = (I - \mathcal{H}) D^{\beta}_{H} w_q$ in (A.19), we get

$$|\mathcal{A}(D^{\beta}_{H} w_q, (I - \mathcal{H}) D^{\beta}_{H} w_q)| \le C \|q\|_{H^{m-1/2, m/2-1/4}(\Gamma, \mathbb{R})} \|D^{\beta}_{H} w_q\|_{H^{1,1/2}(Y, \mathbb{R})}.$$

(Note by our choices of H and σ_2 that $v \in H^{1,1/2}_{0}(Y_{\sigma_2}, \mathbb{R})$.) Hence, by (A.6), we deduce

$$\|D^{\beta}_{H} w_q\|_{H^{1,1/2}} \le C(m) \|q\|_{H^{m-1/2, m/2-1/4}(\Gamma, \mathbb{R})},$$

which by Lemma A.1 proves the theorem. □

B Proof of Theorem 5.7

In this appendix, we give the proof of Theorem 5.7. To recall, we must show that the solution u to the Dirichlet problem

$$\begin{aligned} \frac{\partial u}{\partial t} - \Delta u &= 0 \quad \text{on} \quad \mathbb{R}^3 \backslash \Gamma \times \mathbb{R}, \\ u &= g \quad \text{on} \quad \Gamma \times \mathbb{R}, \end{aligned}$$

satisfies the interior and exterior estimates

$$\|u\|_{H^{m+1,m/2+1/2}(\Omega,\mathbb{R})} \leq C\|g\|_{H^{m+1/2,m/2+1/4}(\Gamma,\mathbb{R})} \quad \text{for each} \quad m \in \mathbb{N}, \tag{B.1}$$

and

$$\|u\|_{\mathcal{W}^{m+1,m/2+1/2}(\Omega^c,\mathbb{R})} \leq C\|g\|_{H^{m+1/2,m/2+1/4}(\Gamma,\mathbb{R})} \quad \text{for each} \quad m \in \mathbb{N}. \tag{B.2}$$

Observe that the regularity estimates (B.1–B.2) do not say anything about the solution u over $\mathbb{R}^3 \times \mathbb{R}$. Therefore, we can separately study the Dirichlet problem over the interior $\Omega \times \mathbb{R}$ and the exterior $\Omega^c \times \mathbb{R}$. We will concentrate on the exterior problem

$$\frac{\partial u}{\partial t} - \Delta u = 0 \quad \text{on} \quad \Omega^c \times \mathbb{R}, \tag{B.3}$$

$$u = g \quad \text{on} \quad \Gamma \times \mathbb{R}. \tag{B.4}$$

The proof will use localization. Henceforth, we fix $m \in \mathbb{N}$ and $g \in H^{m+1/2,m/2+1/4}(\Gamma,\mathbb{R})$. For convenience, we will write

$$\|g\|_{m,\Gamma} = \|g\|_{H^{m+1/2,m/2+1/4}(\Gamma,\mathbb{R})}. \tag{B.5}$$

Continuing with previous notation, let $(O_j, \phi_j, \zeta_j)_1^M$ denote a chart of Γ. We append to this chart an interior set O_0 such that O_0, O_1,..., O_M covers $\bar{\Omega}$ and a function $\zeta_0 \in \mathcal{D}(O_0)$ such that $\{\zeta_j(x)\}_0^M$ forms a partition of unity with respect to this cover. We then set

$$O_{M+1} = \mathbb{R}^3 \backslash \bigcup_{j=0}^{M} O_j,$$

and define

$$\zeta_{M+1}(x) = 1 - \sum_{j=0}^{M} \zeta_j(x), \quad x \in \mathbb{R}^3.$$

(Again, note that the functions $\{\zeta_j(x)\}_0^{M+1}$ form a partition of unity over $\mathbb{R}^3$.)

Now, let u denote the solution to (B.3–B.4). To study u, we will consider each of the functions $\zeta_j u$, for integers j between 1 and $M+1$. They are connected by

$$u(x,t) = \sum_{j=1}^{M+1} \zeta_j(x) u(x,t), \qquad x \in \Omega^c,\ t \in \mathbb{R}. \tag{B.6}$$

To estimate the functions $\zeta_j u$, we shall use a standard inductive argument. Specifically, assume that the solution u to (B.4) satisfies

$$u \in \mathcal{W}^{k,k/2}(\Omega^c, \mathbb{R}), \tag{B.7}$$

for some $k \in \mathbb{N}$ with $k \le m$. We will then show that each $\zeta_j u$ satisfies

$$\|\zeta_j u\|_{\mathcal{W}^{k+1,k/2+1/2}(\Omega^c,\mathbb{R})} \le C\left\{\|g\|_{m,\Gamma} + \|u\|_{\mathcal{W}^{k,k/2}(\Omega^c,\mathbb{R})}\right\}. \tag{B.8}$$

Clearly, because of (B.6), equation (B.8) implies that

$$\|u\|_{\mathcal{W}^{k+1,k/2+1/2}(\Omega^c,\mathbb{R})} \le C\left\{\|g\|_{m,\Gamma} + \|u\|_{\mathcal{W}^{k,k/2}(\Omega^c,\mathbb{R})}\right\}, \tag{B.9}$$

for any $k \in \mathbb{N}$ with $k \le m$. Noting that (B.7) is true for $k = 0$, induction on k in (B.9) shows that

$$\|u\|_{\mathcal{W}^{m+1,m/2+1/2}(\Omega^c,\mathbb{R})} \le C\|g\|_{m,\Gamma} \quad \text{for any} \quad m \in \mathbb{N}. \tag{B.10}$$

The estimates (B.8) will be proven based the differential equation

$$\Lambda(\zeta_j u)(x,t) = 2\nabla u(x,t)\nabla\zeta_j(x) + u(x,t)\Delta\zeta_j(x), \qquad (x,t) \in \Omega^c \times \mathbb{R}, \tag{B.11}$$

$$\zeta_j u(x,t) = \begin{cases} \zeta_j(x) g(x,t) & j = 1,2,\dots M \\ 0 & j = M+1 \end{cases}, \qquad x \in \Gamma,\ t \in \mathbb{R}.$$

solved by $\zeta_j u$. In the next theorem, we show (B.8) for $j = M+1$.

Theorem B.1 *For any integer $k \in \mathbb{N}$, let $u \in \mathcal{W}^{k,k/2}(\Omega^c, \mathbb{R})$ satisfy*

$$\Lambda u(x,t) = 0, \qquad x \in \Omega^c,\ t \in \mathbb{R}, \tag{B.12}$$

and let $\theta(x) \in \mathcal{D}(\mathbb{R}^3)$ satisfy $\theta(x) = 1$ on $\bar{\Omega}$. Then, there exists a positive constant C which depends only on k and θ such that

$$\|(1-\theta)u\|_{\mathcal{W}^{k+1,k/2+1/2}(\Omega^c,\mathbb{R})} \le C\|u\|_{\mathcal{W}^{k,k/2}(\Omega^c,\mathbb{R})}. \tag{B.13}$$

Proof: We apply the heat operator Λ to $(1-\theta)u$. We get

$$\begin{aligned} \Lambda(1-\theta)u &= (1-\theta)\Lambda u + F \\ &= F, \qquad\qquad \text{(by (B.12))}, \end{aligned} \tag{B.14}$$

where

$$F(x,t) = 2\nabla u(x,t)\nabla\theta(x) + u(x,t)\Delta\theta(x), \qquad x \in \Omega^c,\ t \in \mathbb{R}.$$

Since $\theta = 1$ on $\bar{\Omega}$, it follows that $F = 0$ on $\bar{\Omega} \times \mathbb{R}$. Because $(1-\theta)u$ also vanishes over this set, we can extend both sides of equation (B.14) by zero to $\bar{\Omega}$. We get

$$\Lambda(1-\theta)(x,t) = F(x,t). \qquad x \in \mathbb{R}^3,\ t \in \mathbb{R}.$$

To conclude the proof, we will verify the inclusion

$$F(x,t) \in H^{k-1,k/2-1/2}(\mathbb{R}^3,\mathbb{R}) \cap \mathcal{W}^{-1,-1/2}. \tag{B.15}$$

and apply Theorem 5.6. (This theorem showed that

$$\Lambda : \mathcal{W}^{k+1,k/2+1/2} \to H^{k-1,k/2-1/2}(\mathbb{R}^3,\mathbb{R}) \cap \mathcal{W}^{-1,-1/2},$$

is an isomorphism.) The main observation to showing (B.15) is to note that the support of $F(x,t)$ is contained in some cylinder $B_R \times \mathbb{R}$ of finite radius R. It is easily checked that any $H^{k-1,k/2-1/2}(\mathbb{R}^3,\mathbb{R})$ function which is supported in this fashion also belongs to $\mathcal{W}^{-1,-1/2}$.

Hence, we must verify that

$$2\nabla u\nabla\theta + u\Delta\theta \in H^{k-1,k/2-1/2}(\mathbb{R}^3,\mathbb{R}). \tag{B.16}$$

But, by definition of $\mathcal{W}^{k,k/2}(\Omega^c,\mathbb{R})$, we have the existence of positive constant C_1 such that

$$\| \, |\nabla u| \, \|_{H^{k-1,k/2-1/2}(\Omega^c,\mathbb{R})} \leq C_1 \|u\|_{\mathcal{W}^{k,k/2}(\Omega^c,\mathbb{R})}, \qquad u \in \mathcal{W}^{k,k/2}(\Omega^c,\mathbb{R}).$$

Similarly, since $\mathcal{W}^{k,k/2}(\Omega^c,\mathbb{R})$ and $H^{k,k/2}(\Omega^c,\mathbb{R})$ differ only in their behavior near infinity, it follows that there exists a positive constant $C_2(\theta)$ such that

$$\|u\Delta\theta\|_{H^{k,k/2}(\mathbb{R}^3,\mathbb{R})} \leq C(\theta)\|u\|_{\mathcal{W}^{k,k/2}(\Omega^c,\mathbb{R})}.$$

Hence, (B.16) holds. □

It remains to consider (B.11) for the indices j which lie between 1 and M. Since the supports of the functions $\zeta_j u$ are bounded in this case, we add $\zeta_j u$ to each side of (B.11). We get

$$\begin{aligned} \Lambda(\zeta_j u) + \zeta_j u &= F_j(x,t), \qquad (x,t) \in \Omega^c \times \mathbb{R}, \\ \zeta_j u &= \zeta_j g(x,t), \qquad (x,t) \in \Gamma \times \mathbb{R}, \end{aligned} \tag{B.17}$$

where

$$\begin{aligned} F_j(x,t) &= -2\nabla u(x,t)\nabla\zeta_j(x) - u(x,t)\Delta\zeta_j(x) + \zeta_j(x)u(x,t), \\ &\qquad (x,t) \in \Omega^c \times \mathbb{R}, \quad j = 1,\ldots,M. \end{aligned}$$

The way to study the parabolic problems represented in (B.17) is to map them into equivalent problems which are posed over the half space $Y_+ \times \mathbb{R}$, where

$$Y_+ = \{(y_1,y_2,y_3) \colon |y_i| < 1, \, i = 1,2, \, 0 < y_3 < 1\}.$$

Before we do this, it is convenient to reduce (B.17) to an equivalent Dirichlet problem with homogeneous boundary condition. Let $u_g(x,t) \in \mathcal{W}^{m+1,m/2+1/2}$ denote any extension of g such that

$$u_g(x,t) = g(x,t), \qquad (x,t) \in \Gamma \times \mathbb{R}.$$

We then define $U_j = \zeta_j u - \zeta_j u_g$. This function satisfies

$$\begin{aligned} \Lambda(U_j) + U_j &= f_j(x,t), \qquad (x,t) \in \mathbb{R}^3 \times \mathbb{R}, \\ U_j &= 0, \qquad (x,t) \in \Gamma \times \mathbb{R}, \end{aligned} \tag{B.18}$$

where

$$f_j(x,t) = F_j(x,t) - \Lambda\left(\zeta_j u_g\right)(x,t), \qquad (x,t) \in \Omega^c \times \mathbb{R}.$$

To map (B.18) to the half space $Y_+ \times \mathbb{R}$, let

$$w_j(y,t) = \phi_j^*(U_j)(y,t), \qquad (y,t) \in Y_+ \times \mathbb{R}.$$

This leads to

$$\begin{array}{rcll} \frac{\partial w_j}{\partial t}(y,t) - Lw_j(y,t) + w_j(y,t) &=& \phi_j^*(f_j)(y,t), & (y,t) \in Y_+ \times \mathbb{R}, \\ w_j(y,t) &=& 0, & (y,t) \in Y_0 \times \mathbb{R}, \end{array} \tag{B.19}$$

where L denotes the uniformly strongly elliptic differential operator

$$Lw(y,t) = \sum_{i,j=1}^{3} a_{i,j}(y) \frac{\partial^2 w}{\partial y_i \partial y_j}(y,t), \qquad (y,t) \in Y \times \mathbb{R}.$$

(In particular, the matrix of coefficients $\{a_{i,j}(y)\}$ is positive definite, uniformly for $y \in Y$.) Let

$$L_0 w(y,t) = \sum_{i,j=1}^{3} a_{i,j}(0) \frac{\partial^2 w}{\partial y_i \partial y_j}, \qquad (y,t) \in Y_+ \times \mathbb{R}^3. \tag{B.20}$$

The mapping properties of the parabolic operator $I + \Lambda_0 := \partial/\partial t - L_0 + I$ are well understood [25, Chapter 4]. We state them in a lemma.

Lemma B.2 *For each $k \in \mathbb{N}$, let $H_0^{k-1,k/2-1/2}(Y_+,\mathbb{R})$ denote the closure of $\mathcal{D}(Y_+ \times \mathbb{R})$ in the $H^{k-1,k/2-1/2}(Y_+,\mathbb{R})$ norm. For any $f \in H_0^{k-1,k/2-1/2}(Y_+,\mathbb{R})$, there exists a unique solution $w \in H_0^{k+1,k/2+1/2}(Y_+,\mathbb{R})$ to*

$$\frac{\partial w}{\partial t} - L_0 w + w = f \quad on \quad Y_+ \times \mathbb{R}, \tag{B.21}$$

$$w = 0 \quad on \quad Y_0 \times \mathbb{R}, \tag{B.22}$$

which satisfies

$$\|w\|_{H^{k+1,k/2+1/2}(Y_+,\mathbb{R})} \le C(k) \|f\|_{H^{k-1,k/2-1/2}(Y_+,\mathbb{R})}.$$

Set $\Lambda_0 = \partial/\partial t - L_0$ and let $L_1 = L - L_0$, so that

$$\frac{\partial}{\partial t} - L + I = (\Lambda_0 + I) + L_1.$$

To deduce the mapping properties of the operator $I+\Lambda_0+L_1$, we shall show that the operator L_1 amounts in some sense to a small perturbation of the operator $I+\Lambda_0$. The next theorem is the crucial step. For brevity, we shall henceforth use the abbreviations

$$\|u\|_{k,Y_+,\mathbb{R}} = \|u\|_{H^{k,k/2}(Y_+,\mathbb{R})}, \quad \text{and} \quad \|u\|_{k,\mathbb{R}^3_+,\mathbb{R}} = \|u\|_{H^{k,k/2}(\mathbb{R}^3_+,\mathbb{R})},\ k \in \mathbb{Z}.$$

Theorem B.3 *Let $k \in \mathbb{N}$ and $f \in H_0^{k,k/2}(Y_+,\mathbb{R})$. Given any $\epsilon > 0$, there exists some $\delta > 0$ which depends on ϵ and k but not on f, such that supp $f \subset B_\delta \times \mathbb{R}$ implies that*

$$\|L_1(I+\Lambda_0)^{-1} f\|_{k,Y_+,\mathbb{R}} \leq \epsilon \|f\|_{k,Y_+,\mathbb{R}}. \tag{B.23}$$

We of course mean $(I+\Lambda_0)^{-1}$ in the sense of Lemma B.2. We postpone the proof for the moment to show how we will use it. Set

$$z_j = (I+\Lambda_0)\zeta_j u.$$

and by (B.19), we have

$$\left[I + L_1(I+\Lambda_0)^{-1}\right] z_j = \phi_j^*\left(f_j\right).$$

and thus

$$\begin{aligned} \| \left[I + L_1(I+\Lambda_0)^{-1}\right] z_j\|_{k-1,Y_+,\mathbb{R}_+} &\leq \|\phi_j^*(f_j)\|_{k,Y_+,\mathbb{R}_+} \\ &\leq C\|f_j\|_{\mathcal{W}^{k,k/2}(\Omega^c,\mathbb{R})} \\ &\leq C\|u\|_{\mathcal{W}^{k+1,k/2+1/2}(\Omega^c,\mathbb{R})}. \end{aligned} \tag{B.24}$$

Now, by Lemma B.2,

$$\|\zeta_j u\|_{\mathcal{W}^{k+1,k/2+1/2}(\Omega^c,\mathbb{R})} \leq C_0(k)\|z_j\|_{k-1,\mathbb{R}^3,\mathbb{R}}. \qquad k \in \mathbb{N}, \tag{B.25}$$

Since the support of the cutoffs functions ζ_j may be chosen sufficiently small and because

$$\text{supp } z_j \subset \text{ supp}(\ \zeta_j u),$$

we may apply Theorem B.3 and the triangle inequality in (B.24) to deduce the existence of some positive constant C_1 which depends on k and ζ_j such that

$$\|z_j\|_{k,Y_+,\mathbb{R}} \leq C_1 \, \| \, [I + L_1(I+\Lambda_0)^{-1}] z_j \, \|_{k,Y_+,\mathbb{R}}.$$

Combining this with (B.25) yields the desired inequality

$$\|\zeta_j u\|_{\mathcal{W}^{k+2,k/2+1}(\Omega^c,\mathbb{R})} \leq C(k)\|u\|_{\mathcal{W}^{k+1,k/2+1/2}(\Omega^c,\mathbb{R})}.$$

It remains to prove Theorem B.3. We need two lemmas. In these lemmas, we use the notation

$$B_\delta^+ = \{y = (y_1, y_2, y_3) \in \mathbb{R}^3_+ \,:\, y_3 \geq 0,\, |y| \leq \delta\}, \qquad \delta > 0.$$

Lemma B.4 *Let $k \in \mathbb{N}$ and $\theta \in H^{k+1}(\mathbb{R}^3_+)$. Given any $\epsilon > 0$, there exists some $\delta > 0$ which depends on ϵ and k but not on θ, such that supp $\theta \subset B_\delta^+$ implies that*

$$\|\theta\|_{H^k(\mathbb{R}^3_+)} \leq \epsilon \|\theta\|_{H^{k+1}(\mathbb{R}^3_+)}.$$

We refer the reader to [44, p. 256] for the proof and state an important corollary to this result.

Corollary B.5 *Let $k \in \mathbb{N}$ and $\theta \in H^{k+1,k/2+1/2}(\mathbb{R}^3_+, \mathbb{R})$. Given any $\epsilon > 0$, there exists some $\delta > 0$ which depends on ϵ and k but not on θ, such that supp $\theta \subset B_\delta^+ \times \mathbb{R}$ implies that*

$$\|\theta\|_{H^{k,k/2}(\mathbb{R}^3_+,\mathbb{R})} \leq \epsilon \|\theta\|_{H^{k+1,k/2+1/2}(\mathbb{R}^3_+,\mathbb{R})}. \tag{B.26}$$

Proof: Let $\epsilon > 0$, $k \in \mathbb{N}$, and $\theta \in H^{k,k/2}(\mathbb{R}^3_+, \mathbb{R})$ be given. The key to the corollary is to note by interpolation theory that there exists some positive constant C_0 such that

$$\|\theta\|_{H^{k/2}(\mathbb{R},H^1(\mathbb{R}^3_+))} \leq C_0 \|\theta\|_{H^{k+1,k/2+1/2}(\mathbb{R}^3_+,\mathbb{R})}, \qquad k \in \mathbb{N}.$$

Now, by Lemma B.4, there exists some $\delta_1 > 0$ such that

$$\|\theta\|_{L^2(\mathbb{R},H^k(\mathbb{R}^3_+))} \leq \frac{\epsilon}{2C_0} \|\theta\|_{L^2(\mathbb{R},H^{k+1}(\mathbb{R}^3_+))},$$

if the support of θ is contained in $B_{\delta_1}^+ \times \mathbb{R}$. Analogously, there exists some $\delta_2 > 0$ such that

$$\|\theta\|_{H^{k/2}(\mathbb{R},L^2(\mathbb{R}^3_+))} \leq \frac{\epsilon}{2C_0} \|\theta\|_{H^{k/2}(\mathbb{R},H^1(\mathbb{R}^3_+))}, \qquad k \in \mathbb{N},$$

if the support of θ is contained in $B_{\delta_2}^+ \times \mathbb{R}$. The corollary follows by setting $\delta = \min(\delta_1, \delta_2)$ and restricting the support of θ to lie in $B_\delta^+ \times \mathbb{R}$. □

Lemma B.6 *Let $k \in \mathbb{N}$ and $\theta \in H^{k+2,k/2+1}(\mathbb{R}^3_+, \mathbb{R})$. Given any $\epsilon > 0$, there exists some $\delta > 0$ which depends on k and ϵ but not on θ, such that supp $\theta \subset B^+_\delta \times \mathbb{R}$ implies that*

$$\|L_1\theta\|_{k,\mathbb{R}^3_+,\mathbb{R}} \leq \epsilon \|\theta\|_{k+2,\mathbb{R}^3_+,\mathbb{R}} \tag{B.27}$$

Proof: Fix $k \in \mathbb{N}$ and let $\theta \in H^{k+1,k/2+1/2}(\mathbb{R}^3, \mathbb{R})$ be given. Without loss of generality, we can assume that the support of θ is contained in the half cylinder $B^+_\delta \times \mathbb{R}$ for some $0 < \delta < 1$. From the definition of the operator L_1, we shall derive two inequalities.

First, by direct differentiation and the product rule, there exists a positive constant $C_1(k)$ such that

$$\begin{aligned} \|L_1\theta\|_{L^2(\mathbb{R},H^k(\mathbb{R}^3_+))} \quad &\leq \quad \left\{\max_{i,j} \|a_{i,j}(y) - a_{i,j}(0)\|_{L^\infty(B^+_\delta)}\right\} \|\theta\|_{L^2(\mathbb{R},H^{k+2}(\mathbb{R}^3_+))} \\ &\quad + \; C_1(k)\|\theta\|_{L^2(\mathbb{R},H^{k+1}(\mathbb{R}^3_+))}. \end{aligned} \tag{B.28}$$

Analogously, we have

$$\|L_1\theta\|_{H^{k/2}(\mathbb{R},L^2(\mathbb{R}^3_+))} \leq \left(\max_{i,j} \|a_{i,j}(y) - a_{i,j}(0)\|_{L^\infty(B^+_\delta)}\right) \|\theta\|_{H^{k/2}(\mathbb{R},H^2(\mathbb{R}^3_+))}.$$

Since

$$\|\theta\|_{H^{k/2}(\mathbb{R},H^2(\mathbb{R}^3_+))} \leq C_2(k)\|\theta\|_{k+2,\mathbb{R}^3_+,\mathbb{R}},$$

by interpolation theory, we have

$$\|L_1\theta\|_{H^{k/2}(\mathbb{R},L^2(\mathbb{R}^3_+))} \leq C_2(k) \left(\max_{i,j} \|a_{i,j}(y) - a_{i,j}(0)\|_{L^\infty(B^+_\delta)}\right) \|\theta\|_{k+2,\mathbb{R}^3_+,\mathbb{R}}. \tag{B.29}$$

Hence, by combining (B.28–B.29), we have

$$\begin{aligned} \|L_1\theta\|_{k,\mathbb{R}^3_+,\mathbb{R}} \quad &\leq \quad C_3(k)\Bigg\{\left(\max_{i,j} \|a_{i,j}(y) - a_{i,j}(0)\|_{L^\infty(B^+_\delta)}\right) \|\theta\|_{k+2,\mathbb{R}^3_+,\mathbb{R}} \\ &\quad + \; \|\theta\|_{k+1,\mathbb{R}^3_+,\mathbb{R}}\Bigg\}, \end{aligned} \tag{B.30}$$

for some positive constant $C_3(k)$.

Now, the continuity of the coefficients $a_{i,j}$ implies that for any given $\epsilon > 0$, there exists some $\delta_1 > 0$ such that

$$C_3(k)\left\{\max_{i,j} \|a_{i,j}(y) - a_{i,j}(0)\|_{L^\infty(B^+_{\delta_1})}\right\} \leq \frac{\epsilon}{2}. \tag{B.31}$$

Analogously, by Corollary B.5, there exists some $\delta_2 > 0$ such that

$$C_3(k)\|\theta\|_{k+1,\mathbb{R}^3_+,\mathbb{R}} \le \frac{\epsilon}{2}\|\theta\|_{k+2,\mathbb{R}^3_+,\mathbb{R}}, \tag{B.32}$$

for all θ supported in $B^+_{\delta_2} \times \mathbb{R}$. The lemma follows by combining (B.31) with (B.32) and choosing $\delta < \min(\delta_1, \delta_2)$. □

Proof of Theorem B.3 : Fix $k \in \mathbb{N}$ and let $f \in H_0^{k,k/2}(Y_+, \mathbb{R})$ have support contained in $B^+_\delta \times \mathbb{R}$ for some $0 < \delta < 1$. Choose any $\rho \in \mathcal{D}(Y_+)$ with $\rho = 1$ on the support of f. We shall now show the existence of some positive constant C_0 which depends on k and ρ but not on f, such that

$$\|(1-\rho)(I+\Lambda_0)^{-1}f\|_{k+3,Y_+,\mathbb{R}} \le C_0(\rho,k)\|f\|_{k,Y_+\mathbb{R}}. \tag{B.33}$$

Note that it is the difference of 3 between the subscripts of the norms as opposed to 2 which makes this a non-trivial statement.

To prove it, observe that

$$\begin{aligned}(I+\Lambda_0)(1-\rho)(I+\Lambda_0)^{-1}f &= f-(I+\Lambda_0)\rho(I+\Lambda_0)^{-1}f \\ &= (f-\rho f)+G,\end{aligned}$$

where

$$G = \rho f - (I+\Lambda_0)\rho(I+\Lambda_0)^{-1}f.$$

But, since ρ equals one on the support of f, it follows that

$$(I+\Lambda_0)(1-\rho)(I+\Lambda_0)^{-1}f = G. \tag{B.34}$$

Now, set $w_0 = (I+\Lambda_0)^{-1}f$. Because

$$\begin{aligned}(I+\Lambda_0)\rho w_0 - \rho(I+\Lambda_0)w_0 &= \sum_{i,j} a_{i,j}(0)\frac{\partial w_0}{\partial y_i}\frac{\partial \rho}{\partial y_j} \\ &\quad + \sum_{i,j} a_{i,j}(0) w_0 \frac{\partial^2 \rho}{\partial y_i \partial y_j},\end{aligned}$$

it follows that

$$\begin{aligned}G(y,t) &= \sum a_{i,j}(0)\frac{\partial w_0}{\partial y_i}\frac{\partial \rho}{\partial y_j} \\ &\quad + \sum_{i,j} a_{i,j}(0) w_0(y,t)\frac{\partial^2 \rho}{\partial y_i \partial y_j},\end{aligned}$$

for each $(y,t) \in Y_+ \times \mathbb{R}$. Hence,

$$\|G\|_{k,Y_+,\mathbb{R}} \leq C_1(\rho,k)\|w_0\|_{k+1,Y_+,\mathbb{R}}. \tag{B.35}$$

We now applying the operator $(I+\Lambda_0)^{-1}$ to both sides of (B.34). Doing this and then taking norms, we get

$$\begin{aligned} \|(1-\theta)(I+\Lambda_0)^{-1}f\|_{k+3,Y_+,\mathbb{R}} &= \|(I+\Lambda_0)^{-1}G\|_{k+3,Y_+,\mathbb{R}} \\ &\leq C_2(k)\|G\|_{k+1,Y_+,\mathbb{R}} \\ &\leq C_3(\rho,k)\|(I+\Lambda_0)^{-1}f\|_{k+2,Y_+,\mathbb{R}} \\ &\leq C_0(\rho,k)\|f\|_{k,Y_+,\mathbb{R}}, \end{aligned} \tag{B.36}$$

which verifies (B.33).

To complete the proof, let ϵ_1 and ϵ_2 denote two positive numbers to be determined. To estimate $L_1(I+\Lambda_0)^{-1}f$, write

$$L_1(I+\Lambda_0)^{-1}f = L_1(1-\rho)(I+\Lambda_0)^{-1}f + L_1\rho(I+\Lambda_0)^{-1}f,$$

and take norms. This yields

$$\begin{aligned} \|L_1(I+\Lambda_0)^{-1}f\|_{k,Y_+,\mathbb{R}} &\leq \|L_1(1-\rho)(I+\Lambda_0)^{-1}f\|_{k,Y_+,\mathbb{R}} \\ &\quad + \|L_1\rho(I+\Lambda_0)^{-1}f\|_{k,Y_+,\mathbb{R}} \\ &\leq C_4(k)\|(1-\rho)(I+\Lambda_0)^{-1}f\|_{k+2,Y_+,\mathbb{R}} \\ &\quad + \|L_1\theta(I+\Lambda_0)^{-1}f\|_{k-1,Y_+,\mathbb{R}}. \end{aligned}$$

We first apply Lemma B.6 to this inequality to get

$$\begin{aligned} \|L_1(I+\Lambda_0)^{-1}f\|_{k,Y_+,\mathbb{R}} &\leq C_1(k)\|(1-\rho)(I+\Lambda_0)^{-1}f\|_{k+2,Y_+,\mathbb{R}} \\ &\quad + \epsilon_2\|\rho(I+\Lambda_0)^{-1}f\|_{k+2,Y_+,\mathbb{R}}. \end{aligned} \tag{B.37}$$

By then using

$$\begin{aligned} \|\rho(I+\Lambda_0)^{-1}f\|_{k+2,Y_+,\mathbb{R}} &\leq \|(I+\Lambda_0)^{-1}f\|_{k+2,Y_+,\mathbb{R}} \\ &\quad + \|(1-\rho)(I+\Lambda_0)^{-1}f\|_{k+2,Y_+,\mathbb{R}}, \end{aligned}$$

in (B.37), we have

$$\begin{aligned} \|L_1(I+\Lambda_0)^{-1}f\|_{k,Y_+,\mathbb{R}} &\leq (C_1(k)+\epsilon_2)\|(1-\rho)(I+\Lambda_0)^{-1}f\|_{k+2,Y_+,\mathbb{R}} \\ &\quad + \epsilon_2\|(I+\Lambda_0)^{-1}f\|_{k+2,Y_+,\mathbb{R}}. \end{aligned}$$

To conclude, apply (B.33) to this equation to get

$$\begin{aligned}\|L_1(I+\Lambda_0)^{-1}f\|_{k,Y_+,\mathbb{R}} &\le C_0(k)(C_1(k)+\epsilon_2)\|f\|_{k-1,Y_+,\mathbb{R}} \\ &\quad + \epsilon_2\|(I+\Lambda_0)^{-1}f\|_{k+2,Y_+,\mathbb{R}} \\ &\le C_0(k)(C_1(k)+\epsilon_2)\|f\|_{k-1,Y_+,\mathbb{R}} \\ &\quad + \epsilon_2 C_2(k)\|f\|_{k,Y_+,\mathbb{R}}.\end{aligned}$$

and then apply (B.26) to the first term to deduce

$$\|L_1(I+\Lambda_0)^{-1}f\|_{k,Y_+,\mathbb{R}} \le C(k,\epsilon,\epsilon_2)\|f\|_{k,Y_+,\mathbb{R}}, \tag{B.38}$$

where

$$C = \epsilon_1 C_0(k)(C_1(k)+\epsilon_2) + \epsilon_2 C_2(k).$$

Clearly, this implies the theorem. □

C Solutions to the Heat Equation Over the Unit Circle

In this appendix, we describe how to solve the heat equation

$$\begin{aligned}
\frac{\partial u}{\partial t}(x,t) - \Delta u(x,t) &= 0, && x \in B_1,\ t > 0, \\
u(x,0) &= 0, && x \in B_1, \\
u(x,t) &= g(x,t), && x \in \partial B_1,\ t > 0,
\end{aligned} \tag{C.39}$$

in the unit disk B_1. Again, we will describe the points of B_1 by polar coordinates (r,θ) with θ scaled to lie between 0 and 1. For simplicity, we shall assume that $g(\cdot,0) = 0$.

The first step is transform to a Dirichlet problem where the inhomogeneous condition appears as a forcing function. This is done by letting $\tilde{g}(r,\theta,t)$ denote any extension of the Dirichlet data g to the interior of the circle and then letting $U = u - \tilde{g}$. Though $\tilde{g}$ is obviously non-unique, the best choice appears to be an harmonic extension. In other words, we choose $\tilde{g}$ so that

$$\begin{aligned}
\Delta\tilde{g}(r,\theta,t) &= 0, && r > 0,\ \theta \in [0,1],\ t > 0, \\
\tilde{g}(1,\theta,t) &= g(\theta,t), && \theta \in [0,1],\ t > 0.
\end{aligned}$$

With this choice of $\tilde{g}$, we find that U satisfies the inhomogeneous equation

$$\frac{\partial U}{\partial t}(r,\theta,t) - \Delta U(r,\theta,t) = -\frac{\partial \tilde{g}}{\partial t}(r,\theta,t), \qquad r > 0,\ \theta \in [0,1],\ t > 0, \tag{C.40}$$

with homogeneous initial and boundary conditions.

Problem (C.40) is now in the form where Duhamel's principle (cf., [33, p. 205]) applies. For each $s \geq 0$, let $v(r,\theta,t;s)$ denote the solution to

$$\begin{aligned}
\frac{\partial v}{\partial t}(r,\theta,t;s)\Delta v(r,\theta,t;s) &= 0, && t > s, \\
v(1,\theta,t;s) &= 0, && t > s, \\
v(r,\theta,s;s) &= -\frac{\partial \tilde{g}}{\partial t}(r,\theta,t).
\end{aligned} \tag{C.41}$$

Note that the variable s in (C.41) represents the initial time and is viewed as a parameter. Duhamel's principle states that the solution to (C.40) is given by

$$U(r,\theta,t) = \int_0^t v(r,\theta,t;s)ds.$$

Of course, the transformations used to get to (C.41) apply for any problem, not simply ones over the unit circle. The reason why the circle is special is that (C.41) can be solved using separation of variables. The general form of the solution v is

$$v(r,\theta,t;s) = \sum_{m=-\infty}^{\infty}\sum_{k=1}^{\infty} A_{k,m}(s) J_m(\alpha_{m,k} r) e^{-\alpha_{m,k}^2 t} e^{im\theta}, \qquad \text{(C.42)}$$

where $\alpha_{m,k}$ are the roots of

$$J_m(\alpha_{m,k}) = 0, \qquad m \in \mathbb{Z},\ k \in \mathbb{Z}_+,$$

and $A_{k,m}(s)$ the so called Fourier-Bessel coefficients which are determined by

$$-\frac{\partial g}{\partial t}(r,\theta,s) = \sum_{m=-\infty}^{\infty}\sum_{k=1}^{\infty} A_{k,m}(s) J_m(\alpha_{m,k} r) e^{-\alpha_{m,k}^2 s} e^{im\theta}.$$

Thus,

$$A_{k,m} = \frac{\int_0^{2\pi}\int_0^1 rF_g(r,\theta,s) J_m(\alpha_{m,k} r)\, dr d\theta}{\int_0^1 r\left(J_m(\alpha_{m,k} r)\right)^2\, dr},$$

where we have set $F_g = -\partial g/\partial t$.

Remark: To compute the series in (C.42), the various identities found in [45, p. 502], such as

$$\sum_{k=1}^{\infty}\frac{1}{\alpha_{m,k}^2} = \frac{1}{4(m+1)},$$

and

$$\sum_{m=0}^{\infty}\frac{1}{\alpha_{m,k}^4} = \frac{1}{16(m+1)^2(m+2)}, \qquad m \in \mathbb{Z}_+,$$

were found to be quite useful.

References

[1] D. N. Arnold and W. L. Wendland, "On the Asymptotic Convergence of Collocation Methods," Math. Comp., 41(1983), 349-381.

[2] K. E. Atkinson, *A Survey of Numerical Methods for the Solution of Fredholm Integral equations of the Second Kind,* Soc. Ind. Appl. Math, Philadelphia(1976).

[3] J. P. Aubin, *Approximation of Elliptic Boundary-Value Problems*, Willey Interscience, New York, 1972.

[4] I. Babuska and A. K. Aziz, *The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations* , Academic Press, New York, 1972.

[5] *Boundary Element Techniques* (Eds., C. A. Brebbia, T. Futagami and M. Tanaka), Proceedings of the 5th Int. Conference on BEM, Hiroshima, 1983, Springer-Verlag, Berlin and NY, 1983.

[6] C. Baiocchi and F. Brezzi, "Optimal Error Estimates for Linear Parabolic Problems under Minimal Regularity Assumptions," Calcolo, 20(1983), 143-176.

[7] R. Brown, "Layer Potentials and Boundary Value Problems for the Heat Equation on Lipschitz Cylinders," University of Minnesota, Ph. D. Thesis, 1987.

[8] Y. P. Chang, C. S. Kang and D. J. Chen, "The Use of Fundamental Green's Function for the Solution of Problems of Heat Conduction in Anisotropic Media," Int. J. Heat Transfer, 16, 1903-1918(1973)

[9] P. G. Ciarlet, *The Finite Element Method for Elliptic Problems*, North Holland, Amsterdam, 1978.

[10] M. Costabel, K. Onishi, and W. L. Wendland, "A Boundary Element Collocation Method for the Neumann Problem of the Heat Equation," preprint.

[11] D. Curran, M. Cross, and B. A. Lewis, "A Preliminary Analysis of Boundary Element Methods Applied to Parabolic Partial Differential Equations," in *New Developments in Boundary Element Methods* (C. A. Brebbia, ed.), CML Publications, Southhampton, 1980.

[12] D. Curran and B. A. Lewis, "A Boundary Element Method for the Solution of the Transient Diffusion Equation in Two Dimensions," Appl. Math. Modelling, 10(1986), 107-113.

[13] J. Deny and J. L. Lions, "Les Espaces de Type Beppo Levi," Ann. Inst. Fourier, 5(1953-1954), 305-370.

[14] E. B. Fabes and N. M. Riviere, " Dirichlet and Neumann Problems for the Heat Equation in C^1- Cylinders," Proceedings of Symposia in Pure Mathematics, Vol. XXXV, Part 2 (1979), 179-196.

[15] G. Fairweather, F. J. Rizzo and D. J. Shippy, "Computation of Double Integrals in the Boundary Integral Equation Method," in *Advances in Computer Methods for Partial Differential Equations III*, R. Vichnevetsky and R. S. Stepleman, eds., New Brunswick: IMACS, 1979, 331-334.

[16] G. B. Folland, *Introduction to Partial Differential Equations*, Princeton University Press, Princeton, NJ, 1976.

[17] A.Friedman, *Partial Differential Equations of Parabolic Type* , Prentice-Hall, Englewood-Cliff, NJ (1964)

[18] M. Gevrey, Sur les Equations Aux Derivees Partielles du Type Parabolique," Journal de Math., Ser. 6, Vol. 9, 1913, pp. 305-471.

[19] J. Giroire and J. C. Nedelec, "Numerical Solution of an Exterior Neumann Problem Using a Double Layer Potential, Math. of Comp., 32(1978),973-992.

[20] B. Hanouzet, "Espaces de Sobolev Avec Poids. Application au problem de Dirichlet dans un demi-space," Rendiconti del Seminario Matematico dell'Università di Padova, XLVI(1971), 227-272.

[21] E. Holmgren, "Sur une Application de l'Equation Integrale de M. Volterra," Ark. Mat. Astronom. Fys., Vol. 3, 1907, 1-4.

[22] S. Kaplan, "Abstract Boundary Value Problems for Linear Parabolic Equations," Ann. Scuola. Norm. Sup. Pisa, 20(1966), 395-420.

[23] J. L. Lions, *Equations Differentielles Operationelles et Problems Aux Limits*, Springer Verlag, Berlin, 1961.

[24] J. L. Lions and E. Magnes, *Problems Aux Limites non Homogenous et Applications* , Vol.1, Dunod, Paris, 1968.

[25] J. L. Lions and E. Magnes, *Problems Aux Limites non Homogenous et Applications* , Vol.2, Dunod, Paris, 1968.

[26] E. A. McIntrye Jr., "Boundary Integral Solution of the Heat Equation," Math Comp., 46(1986), 71-79.

[27] G. Miranda, "Integral Equation Solution of the First Initial-Boundary Value Problem for the Heat Equation in Domains with Non-smooth Boundary," Comm. Pure and Appl. Math., XXIII(1970), 757-765.

[28] J. C. Nedelec and J. Planchard, " Une Methode Variationalle d'Elements Finis Pour la Resolution Numerique d'un Probleme Exterieur dans $\mathbb{R}^3$," Rairo R-3, 7(1963), 105-129.

[29] J. C. Nedelec, "Curved Finite Element Methods for the Solution of Singular Integral Equations on Surfaces in $\mathbb{R}^3$," Comput. Methods Appl. Mech. Engrg, 8(1976), 61-80.

[30] H. Okamoto, "Applications of the Fourier Transform to the Boundary Element Method under the Dirichlet Boundary Condition," preprint.

[31] F. W. J. Olver, *Asymptotics and Special Functions*, Academic Press, New York, 1974.

[32] K. Onishi, "Convergence in the Boundary Element Method for the Heat Equation," Tru. Math, 17, 213-225(1981).

[33] Ozisik, M. N., *Heat Conduction,* John Wiley and Sons, New York, 1980.

[34] H. L. G. Pina and J. L. M. Fernandes, "Applications in Transient Heat Conduction." *Chapter 2, Topics in Boundary Element Research,* (Ed. C. A. Brebbia), Vol. 1, pp. 41-58, Springer-Verlag, New York.

[35] W. Pogorzelski, *Integral Equations* , Vol. 1, Pergamon Press, Oxford(1964).

[36] W. Pogorzelski, "Sur la Solution de l'Equation Integrale dans le Probleme de Fourier," Ann. de la Soc. Polon de Math., 24(1951), 56-74.

[37] W. Pogorzelski, "Sur le Probleme de Fourier Generalise," Ann. Polon. Math., 3(1956), 126-141.

[38] W. Pogorzelski, "Probleme Aux Limites Pour l'Equation Parabolique Normale," Ann. Polon. Math., 4(1957), 110-126.

[39] M. N. Le. Roux, "Method D'Elements Finis Pour La Resolution Numerique De Problems Exterieurs en Dimension 2," Rairo. Anal. Num., 1(1977), 27-60.

[40] W. Rudin, *Functional Analysis* , McGraw-Hill Book Company, New York, 1973.

[41] C. Johnson and L. R. Scott, "An Analysis of Quadrature Errors in Second-Kind Boundary Integral Methods,"Preprint, 1987(?).

[42] F. Sgallari, "A Weak Formulation of Boundary Integral Equations for Time Dependent Problems," Appl. Math. Modelling, 9(1985), 295-301.

[43] R. P. Shaw, "An Integral Equation Approach to Diffusion," Int. J. Heat Transfer, 17(1974), 693-699.

[44] F. Treves, *Basic Linear Partial Differential Equations*, Academic Press, New York, 1975.

[45] G. N. Watson, *A Treastise on the Bessel Functions,* Cambridge University Press, London and New York, 1944.

[46] W. L. Wendland, "Asymptotic Convergence of Boundary Element Methods," in *Lectures on the Numerical Solution of Partial Differential Equations*(I. Babuska, T.-P. Liu, and J. Osborn, eds.), Lecture Notes, Vol 20., University of Maryland, College Park, MD., 1981, 435-528.

[47] L. C. Wrobel and C. Brebbia, "Time Dependent Potential Problems," in *Progress in Boundary Element Methods* .