Finite element exterior calculus

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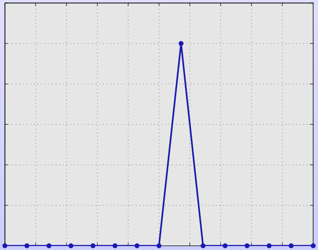
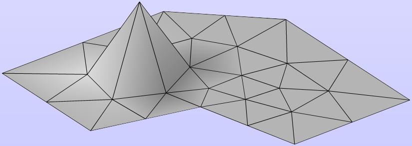
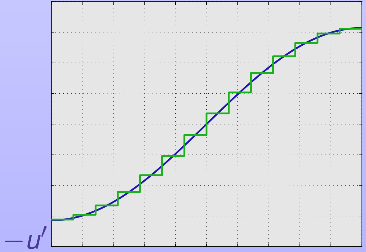
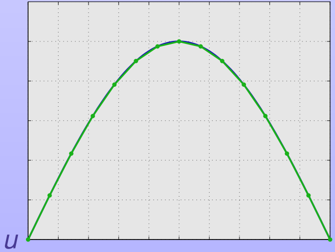
Computational examples

Standard P_1 finite elements for 1D Laplacian

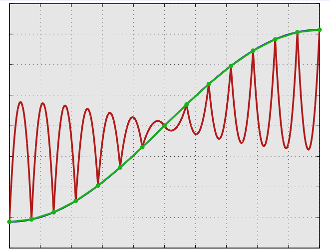




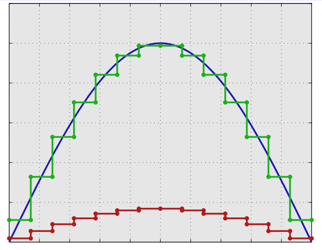




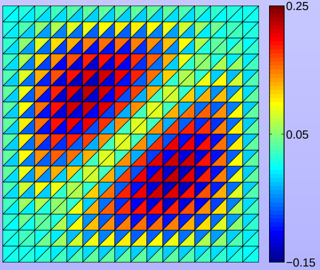
Mixed finite elements for Laplacian



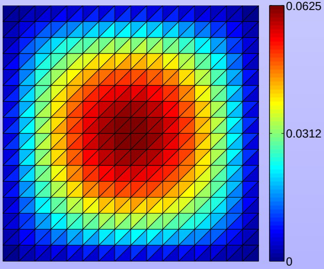
σ



u



u P_1-P_0



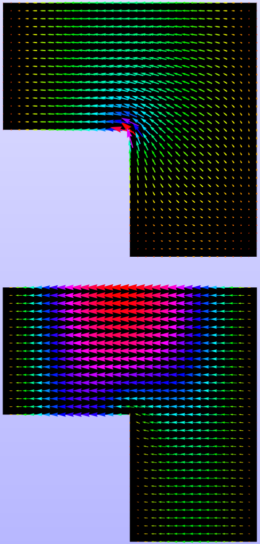
u RT-P_0

Vector Laplacian, L-shaped domain

$$\operatorname{curl}\operatorname{curl} u - \operatorname{grad}\operatorname{div} u = f \text{ in } \Omega$$

$$u \cdot n = 0, \quad \operatorname{curl} u \times n = 0 \text{ on } \partial\Omega$$

$$\int_\Omega (\operatorname{curl} u \cdot \operatorname{curl} v + \operatorname{div} u \operatorname{div} v) = \int_\Omega f \cdot v \quad \forall v$$

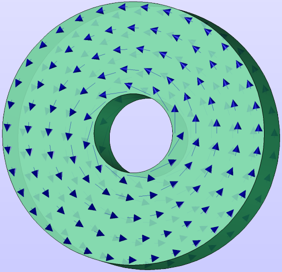


Lagrange finite elements converge nicely
but not to the solution!
(same problem with any conforming FE)

Vector Poisson equation

$$\operatorname{curl}\operatorname{curl} u - \operatorname{grad}\operatorname{div} u = f \quad \text{in } \Omega$$

$$u \cdot n = 0, \ \operatorname{curl} u \times n = 0 \quad \text{on } \partial\Omega$$



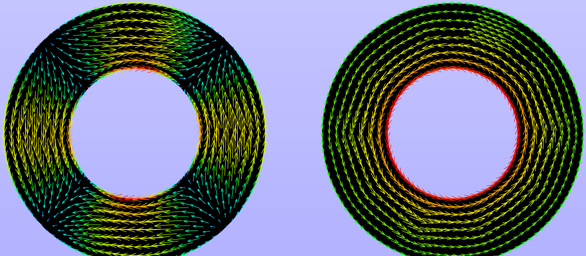
$f \equiv 0$ does not imply $u \equiv 0$:

$$\dim \mathfrak{H} = b_1$$

harmonic forms
(solutions for $f = 0$)

1st Betti number
(number of holes)

$$\operatorname{curl}\operatorname{curl} u - \operatorname{grad}\operatorname{div} u = f \ (\operatorname{mod} \mathfrak{H}), \ u \perp \mathfrak{H}, \ \text{b.c.}$$

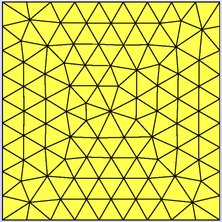


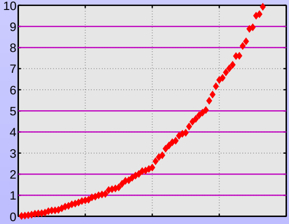
Maxwell eigenvalue problem

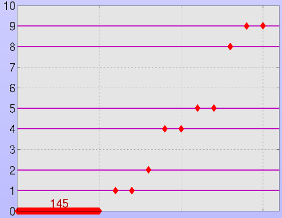
Find $0 \neq u \in H(\mathrm{curl})$ s.t

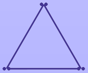
$$\int_\Omega \mathrm{curl}\, u \cdot \mathrm{curl}\, v = \lambda \int_\Omega u \cdot v \quad \forall v \in H(\mathrm{curl})$$

$$\lambda = m^2 + n^2 = 0, 1, 1, 2, 4, 4, 5, 5, 8, \ldots$$

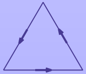








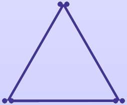
$\mathcal{P}_1$

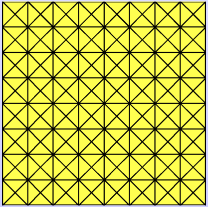


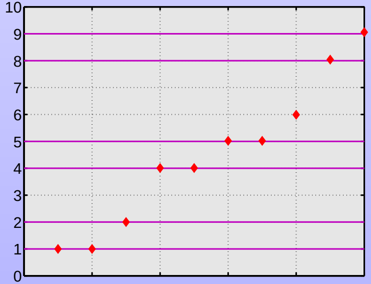
$\mathcal{P}_1^- \Lambda^1$

Maxwell eigenvalue problem, crisscross mesh

$\lambda = m^2 + n^2 = 1, 1, 2, 4, 4, 5, 5, 8, \ldots$





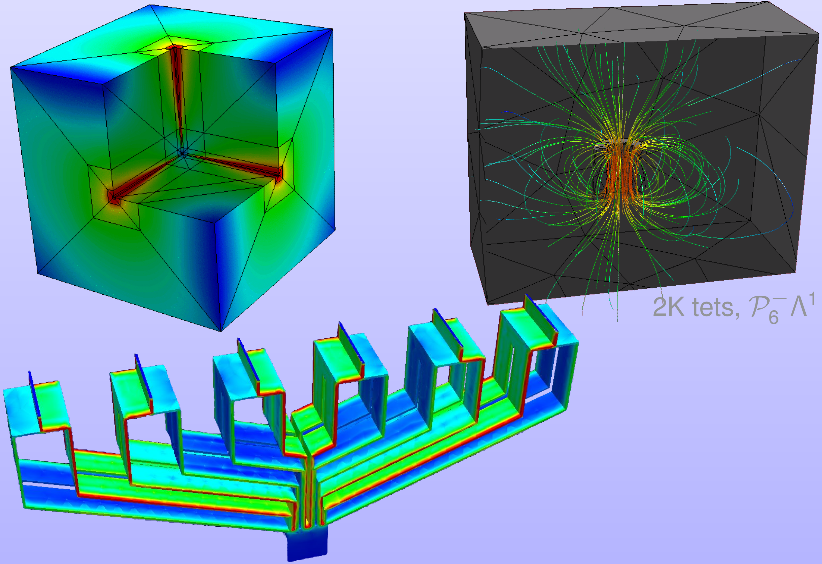


254	574	1022	1598
1.0043	1.0019	1.0011	1.0007
1.0043	1.0019	1.0011	1.0007
2.0171	2.0076	2.0043	2.0027
4.0680	4.0304	4.0171	4.0110
4.0680	4.0304	4.0171	4.0110
5.1063	5.0475	5.0267	5.0171
5.1063	5.0475	5.0267	5.0171
5.9229	5.9658	5.9807	5.9877
8.2713	8.1215	8.0685	8.0438

Boffi-Brezzi-Gastaldi ’99

EM calculations based on the generalized RT elements

Schöberl, Zaglmayr 2006, NGSolve



2K tets, $\mathcal{P}_6^-\Lambda^1$

Homology 101

Chain complexes

- A *chain complex* (V, ∂) is a seq. of vector spaces and linear maps
  $\cdots \to V_{k+1} \xrightarrow{\partial_{k+1}} V_k \xrightarrow{\partial_k} V_{k-1} \to \cdots$ with $\partial^k \circ \partial^{k+1} = 0$.
  typically, non-negative and finite: $V^k = 0$ for $k < 0$ for k large

- In other words, $V = \bigoplus_k V_k$ is a *graded vector space* and
  $\partial : V \to V$ is a *graded linear operator* of degree -1 such that
  $\partial \circ \partial = 0$ $(\partial_k = \partial|_{V_k} : V_k \to V_k)$

- V_k: k-chains
  $\mathfrak{Z}_k = \mathcal{N}(\partial_k)$: k-cycles
  $\mathfrak{B}_k = \mathcal{R}(\partial_{k+1})$: k-boundaries
  $\mathcal{H}_k = \mathfrak{Z}_k / \mathfrak{B}_k$: k-th homology space

- Thus the elements of $\mathcal{H}_k$ are equivalence classes of k-cycles

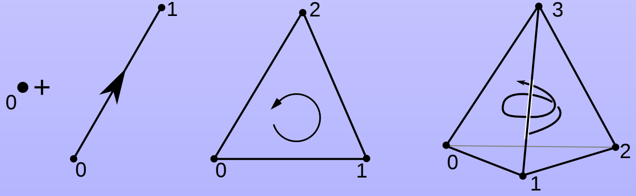
Simplices and simplicial complexes

- A k-simplex in $\mathbb{R}^n$ is the convex hull $f = [x_0, \ldots, x_k]$ of $k+1$ vertices in general position.

- A subset determines a face of f: $[x_{i_0}, \ldots, x_{i_d}]$.

- *Simplicial complex:* A finite set $\mathcal{S}$ of simplices in $\mathbb{R}^n$, such that
  1. Faces of simplices in $\mathcal{S}$ are in $\mathcal{S}$.
  2. If $f \cap g \neq \emptyset$ for $f, g \in \mathcal{S}$, then it is a face of f and of g.

- If we order all vertices of $\mathcal{S}$, then an ordering of the vertices of the simplex determines an *orientation*.



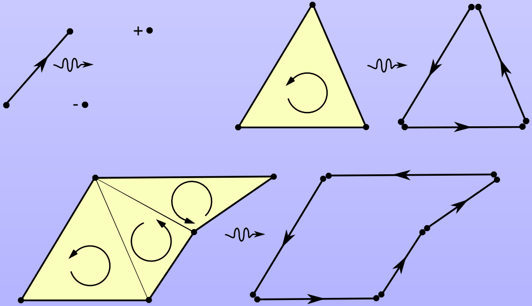
The boundary operator on chains

$\Delta_k(\mathcal{S})$: the set of k-simplices in $\mathcal{S}$

C_k (k-chains): formal linear combinations $c = \sum_{f \in \Delta_k(\mathcal{S})} c_f\, f$

$\partial_k : \Delta_k \to C_{k-1}$: $\partial[x_0, x_1, \ldots, x_k] = \sum_{i=0}^{k} (-1)^i [\ldots, \hat{x}_i, \ldots]$

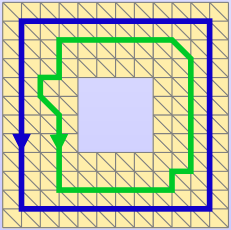
$\partial_k : C_k \to C_{k-1}$: $\partial c = \sum c_f\, \partial f$



The simplicial chain complex

$$0 \to C_n \xrightarrow{\partial} C_{n-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_0 \xrightarrow{\partial} 0$$

$\beta_k := \dim \mathcal{H}_k(C)$ is the kth Betti number



1, 1, 0, 0

1, 1, 0, 0

1, 2, 1, 0

1, 2 , 0, 0

1, 0, 1, 0

Chain maps

$$\begin{tikzcd}
\cdots \arrow[r] & V_{k+1} \arrow[r, "\partial_{k+1}"] \arrow[d, "f_{k+1}"'] & V_k \arrow[r, "\partial_k"] \arrow[d, "f_k"'] & V_{k-1} \arrow[r] \arrow[d, "f_{k-1}"'] & \cdots \\
\cdots \arrow[r] & V'_{k+1} \arrow[r, "\partial'_{k+1}"] & V'_k \arrow[r, "\partial'_k"] & V'_{k-1} \arrow[r] & \cdots
\end{tikzcd}$$

- $f(\mathfrak{Z}) \subset \mathfrak{Z}'$, $f(\mathfrak{B}) \subset \mathfrak{B}'$, so f induces $\bar{f} : \mathcal{H}(V) \to \mathcal{H}(V')$.
- If V' is a subcomplex ($V'_k \subset V_k$ and $\partial' = \partial|_V$), and $fv = v$ for $v \in V'$, we call f a *chain projection*.

Proposition

A chain projection induces a surjection on homology.

Cochain complexes

A cochain complex is like a chain complex but with *increasing* indices.

$$\cdots \to V^{k-1} \xrightarrow{d^{k-1}} V^k \xrightarrow{d^k} V^{k+1} \to \cdots$$

- cocycles $\mathfrak{Z}^k$, coboundaries $\mathfrak{B}^k$, cohomology $\mathcal{H}^k$, ...
- The dual of a chain complex is a cochain complex:

$$\partial_{k+1} : V_{k+1} \to V_k \quad \Longrightarrow \quad \underset{\substack{\uparrow \\ d^k}}{\partial^*_{k+1}} : \underset{\substack{\uparrow \\ V^k}}{V^*_k} \to V^*_{k+1}$$

The de Rham complex for a domain in $\mathbb{R}^n$

1-D: $0 \to C^\infty(\Omega) \xrightarrow{d/dx} C^\infty(\Omega) \to 0$

2-D: $0 \to C^\infty(\Omega) \xrightarrow{\text{grad}} C^\infty(\Omega, \mathbb{R}^2) \xrightarrow{\text{rot}} C^\infty(\Omega) \to 0$

3-D: $0 \to C^\infty(\Omega) \xrightarrow{\text{grad}} C^\infty(\Omega, \mathbb{R}^3) \xrightarrow{\text{curl}} C^\infty(\Omega, \mathbb{R}^3) \xrightarrow{\text{div}} C^\infty(\Omega) \to 0$

n-D: $0 \to \Lambda^0(\Omega) \xrightarrow{d} \Lambda^1(\Omega) \xrightarrow{d} \Lambda^2(\Omega) \xrightarrow{d} \cdots \xrightarrow{d} \Lambda^n(\Omega) \to 0$

The space $\Lambda^k(\Omega) = C^\infty(\Omega, \mathbb{R}^{n\times\cdots\times n}_{\text{skw}})$, the space of smooth *differential k-forms* on Ω.

- *Exterior derivative:* $d^k : \Lambda^k(\Omega) \to \Lambda^{k+1}(\Omega)$
- *Integral of a k-form over an oriented k-simplex:* $\int_f v \in \mathbb{R}$
- *Stokes theorem:* $\int_c du = \int_{\partial c} u, \quad u \in \Lambda^k, c \in C_k$
- All this works on *any smooth manifold*

De Rham's Theorem

- De Rham map:

$$\begin{aligned} \Lambda^k(\Omega) &\longrightarrow C^k(\mathcal{S}) := C_k(\mathcal{S})^* \\ u &\longmapsto \left(c \mapsto \int_c u\right) \end{aligned}$$

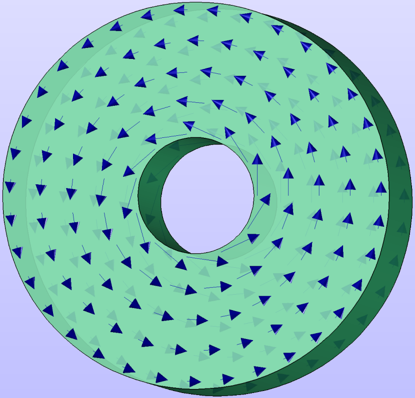
- Stokes theorem says it's a cochain map, so induces a map from de Rham to simplicial cohomology.

$$\begin{tikzcd} \cdots \arrow[r, "d"] & \Lambda^k(\Omega) \arrow[r, "d"] \arrow[d] & \Lambda^{k+1}(\Omega) \arrow[r, "d"] \arrow[d] & \cdots \\ \cdots \arrow[r, "\partial^*"] & C_k^* \arrow[r, "\partial^*"] & C_{k+1}^* \arrow[r, "\partial^*"] & \cdots \end{tikzcd}$$

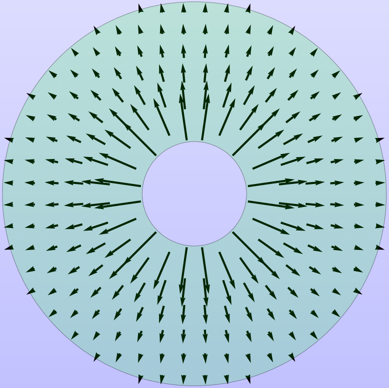
Theorem (De Rham's theorem)

The induced map is an isomorphism on cohomology.

Nonzero cohomology classes



$u = \operatorname{grad}\theta$, $0 \neq \bar{u} \in \mathcal{H}^1$
on cylindrical shell



$u = \operatorname{grad}\frac{1}{r}$, $0 \neq \bar{u} \in \mathcal{H}^2$
on spherical shell

Unbounded operators on Hilbert space

Unbounded operators

- X, Y H-spaces (extensions to Banach spaces, TVSs,...)
- $T : D(T) \to Y$ linear, $D(T) \subseteq X$ subspace (not necessarily closed), T not necessarily bounded
- *Not-necessarily-everywhere-defined-and-not-necessarily-bounded linear operators*
- *Densely defined:* $\overline{D(T)} = X$
- Ex: $X = L^2(\Omega)$, $Y = L^2(\Omega; \mathbb{R}^n)$, $D(T) = H^1(\Omega)$, $Tv = \operatorname{grad} v$ (changing $D(T)$ to $\mathring{H}^1(\Omega)$ gives a *different* example)
- S, T unbdd ops $X \to Y \implies D(S+T) = D(S) \cap D(T)$ (may not be d.d.)
- $X \xrightarrow{S} Y$, $Y \xrightarrow{T} Z$ unbdd ops $\implies D(T \circ S) = \{v \in D(S) \mid Sv \in D(T)\}$
- *Graph norm (and inner product):* $\|v\|^2_{D(T)} := \|v\|^2_X + \|Tv\|^2_Y$, $v \in D(T)$
- *Null space, range, graph:* $\mathcal{N}(T)$, $\mathcal{R}(T)$, $\Gamma(T)$

Closed operators

- T is *closed* if $\Gamma(T)$ is closed in $X \times Y$.
- Equivalent definitions:
  1. If $v_1, v_2, \ldots \in D(T)$ satisfy $v_n \xrightarrow{X} x$ and $Tv_n \xrightarrow{Y} y$ for some $x \in X$ and $y \in Y$, then $x \in D(T)$ and $Tx = y$.
  2. $D(T)$ endowed with the graph norm is complete.
- If $D(T) = X$, then T is closed $\iff$ T is bdd (Closed Graph Thm)

Many properties of bounded operators extend to closed operators. E.g.,

Proposition

Let T be a closed operator X to Y.

1. *$\mathcal{N}(T)$ is closed in X.*
2. $\exists \gamma > 0$ *s.t.* $\|Tx\|_Y \geq \gamma \|x\|_X \iff \mathcal{N}(T) = 0, \mathcal{R}(T)$ *closed*
3. *If* $\dim Y/\mathcal{R}(T) < \infty$, *then* $\mathcal{R}(T)$ *is closed*

Adjoint of a d.d.unbdd operator

Let T be a d.d.unbdd operator $X \to Y$. Define

$$D(T^*) = \{ w \in Y \,|\, \text{the map } v \in D(T) \mapsto \langle w, Tv \rangle_Y \in \mathbb{R} \text{ is bdd in } X\text{-norm} \}$$

For $w \in D(T^*)$ $\quad \exists!\, T^* w \in X$ s.t.

$$\langle T^* w, v \rangle_X = \langle w, Tv \rangle_Y, \quad v \in D(T), \; w \in D(T^*).$$

T^* is a closed operator (even if T is not). Define the rotated graph

$$\tilde{\Gamma}(T^*) = \{ (-T^* w, w) \,|\, w \in D(T^*) \} \subset X \times Y,$$

Then $\Gamma(T)^\perp = \tilde{\Gamma}(T^*)$, $\overline{\Gamma(T)} = \tilde{\Gamma}(T^*)^\perp$.

Proposition

Let T be a closed d.d. operator $X \to Y$. Then

1. *T^* is closed d.d.*
2. *$T^{**} = T$.*
3. $$\mathcal{R}(T)^\perp = \mathcal{N}(T^*), \quad \mathcal{N}(T)^\perp = \overline{\mathcal{R}(T^*)},$$ $$\mathcal{R}(T^*)^\perp = \mathcal{N}(T), \quad \mathcal{N}(T^*)^\perp = \overline{\mathcal{R}(T)}.$$

Closed Range Theorem

Theorem

Let T be a closed d.d.operator $X \to Y$. If $\mathcal{R}(T)$ is closed in Y, then $\mathcal{R}(T^*)$ is closed in X.

Proof.

1. Reduce to case T is surjective.
2. Restrict to orthog comp of $\mathcal{N}(T)$ in $D(T)$ (w/ graph norm). Get bounded linear isomorphism. $\exists$ bounded inverse:

$$\forall y \in Y\ \exists x \in X \text{ s.t. } Tx = y, \quad \|x\|_X \leq c\|y\|_Y$$

3. This implies $\|y\|_Y \leq c\|T^*y\|_X$, $y \in D(T^*)$. □

Grad and div

Assume $\Omega \subset \mathbb{R}^3$ with Lipschitz boundary (so trace theorem holds).

1. Start with $-\operatorname{div} : C_c^\infty \subset L^2(\Omega; \mathbb{R}^3) \to L^2(\Omega)$
2. Its adjoint is grad with domain H^1 (this proves H^1 is complete).
3. The adjoint of $(\operatorname{grad}, H^1)$ is $-\operatorname{div}$ with domain

   $$\mathring{H}(\operatorname{div}) = \{\, w \in H(\operatorname{div}) \mid \gamma_n w := w \cdot n|_{\partial\Omega} = 0 \,\}$$

4. $\operatorname{div} \mathring{H}(\operatorname{div})$ is finite-codimensional so closed. So $\operatorname{grad} H^1$ is closed.

Curl

1. Start with curl : $C_c^\infty \subset L^2(\Omega; \mathbb{R}^3) \to L^2(\Omega; \mathbb{R}^3)$
2. Its adjoint is curl with domain $H(\text{curl})$ (complete).
3. Adjoint of $(\text{curl}, H(\text{curl}))$ with domain

$$\mathring{H}(\text{curl}) = \{ w \in H(\text{curl}) \,|\, \gamma_t w := w \times n|_{\partial\Omega} = 0 \}$$

4. We shall see that curl $H(\text{curl})$ is closed.

Hilbert complexes

Hilbert complexes

Definition

A *Hilbert complex* is a sequence of Hilbert spaces W^k and a sequence of closed d.d.linear operators d^k from W^k to W^{k+1} such that $\mathcal{R}(d^k) \subset \mathcal{N}(d^{k+1})$.

- $V_k = D(d^k)$ H-space with graph norm: $\|v\|^2_{V^k} = \|v\|^2_{W^k} + \|d^k v\|^2_{W^{k+1}}$
- The *domain complex*

$$0 \to V^0 \xrightarrow{d} V^1 \xrightarrow{d} \cdots \xrightarrow{d} V^n \to 0$$

  is a *bounded* Hilbert complex (with less information).
- It is a cochain complex, so it has (co)cycles, boundaries, and homology.
- An H-complex is *closed* if $\mathfrak{B}^k$ is closed in W^k (or V^k).
- An H-complex is *Fredholm* if $\dim \mathcal{H}^k < \infty$.

$$\text{Fredholm} \implies \text{closed}$$

The dual complex

Define $d_k^* : V_k^* \subset W^k \to W^{k-1}$ as the adjoint of $d^{k-1} : V^k \subset W^{k-1} \to W^k$.

It is closed d.d.and, since $\mathcal{R}(d^{k-1}) \subset \mathcal{N}(d^k)$,

$$\mathcal{R}(d_{k+1}^*) \subset \overline{\mathcal{R}(d_{k+1}^*)} = \mathcal{N}(d^k)^\perp \subset \mathcal{R}(d^{k-1})^\perp = \mathcal{N}(d_*^k),$$

so we get a Hilbert *chain* complex with domain complex

$$0 \to V_n^* \xrightarrow{d_n^*} V_{n-1}^* \xrightarrow{d_{n-1}^*} \cdots \xrightarrow{d_1^*} V_0^* \to 0.$$

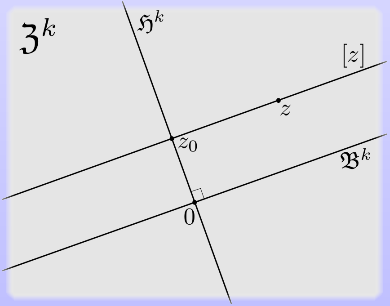
If (W, d) is closed, then (W, d^*) is as well, by the Closed Range Theorem.

From now on we mainly deal with closed H-complexes. . .

Harmonic forms

The Hilbert structure of a *closed* H-complex allows us to identify the homology space $\mathcal{H}^k = \mathfrak{Z}^k/\mathfrak{B}^k$ with a subspace $\mathfrak{H}^k$ of W^k:

$$\mathfrak{H}^k := \mathfrak{Z}^k \cap \mathfrak{B}^{k\perp} = \mathfrak{Z}^k \cap \mathfrak{Z}^*_k = \{u \in V^k \cap V^*_k \mid du = 0,\ d^*u = 0\}.$$



An *H*-complex has the *compactness property* if $V^k \cap V^*_k$ is dense and *compact* in W^k. This implies $\dim \mathfrak{H}^k < \infty$.

compactness property $\implies$ Fredholm $\implies$ closed

Two key properties of closed H-complexes

Theorem (Hodge decomposition)

For any closed Hilbert complex:

$$W^k = \underbrace{\mathfrak{B}^k \oplus \mathfrak{H}^k}_{\mathfrak{Z}^k} \oplus \underbrace{\mathfrak{B}^*_k}_{\mathfrak{Z}^{k\perp}}$$

$$V^k = \overbrace{\mathfrak{B}^k \oplus \mathfrak{H}^k} \oplus \mathfrak{Z}^{k\perp_V}$$

Theorem (Poincaré inequality)

For any closed Hilbert complex, $\exists$ a constant c^P s.t.

$$\|z\|_V \leq c^P \|dz\|, \quad z \in \mathfrak{Z}^{k\perp_V}.$$

L^2 de Rham complex on $\Omega \subset \mathbb{R}^3$

k	W^k	d^k	V^k	d_k^*	V_k^*	$\dim \mathfrak{H}^k$
0	$L^2(\Omega)$	grad	H^1	0	L^2	β_0
1	$L^2(\Omega; \mathbb{R}^3)$	curl	$H(\mathrm{curl})$	$-$ div	$\mathring{H}(\mathrm{div})$	β_1
2	$L^2(\Omega; \mathbb{R}^3)$	div	$H(\mathrm{div})$	curl	$\mathring{H}(\mathrm{curl})$	β_2
3	$L^2(\Omega)$	0	L^2	$-$ grad	$\mathring{H}^1$	0

$$0 \to H^1 \xrightarrow{\mathrm{grad}} H(\mathrm{curl}) \xrightarrow{\mathrm{curl}} H(\mathrm{div}) \xrightarrow{\mathrm{div}} L^2 \to 0$$

$$0 \leftarrow L^2 \xleftarrow{-div} \mathring{H}(\mathrm{div}) \xleftarrow{\mathrm{curl}} \mathring{H}(\mathrm{curl}) \xleftarrow{-\,\mathrm{grad}} \mathring{H}^1 \leftarrow 0$$

The abstract Hodge Laplacian

- $W^{k-1} \underset{d^*}{\overset{d}{\rightleftarrows}} W^k \underset{d^*}{\overset{d}{\rightleftarrows}} W^{k+1} \qquad L := d^*d + dd^* \qquad W^k \xrightarrow{L} W^k$
- $D(L^k) = \{\, u \in V^k \cap V_k^* \mid du \in V_{k+1}^*,\ d^*u \in V^{k-1} \,\}$
- $\mathcal{N}(L^k) = \mathfrak{H}^k, \quad \mathfrak{H}^k \perp \mathcal{R}(L^k)$
- *Strong formulation:* Find $u \in D(L^k)$ s.t. $Lu = f - P_{\mathfrak{H}} f, \quad u \perp \mathfrak{H}$.
- *Primal weak formulation:* Find $u \in V^k \cap V_k^* \cap \mathfrak{H}^{k\perp}$ s.t.

$$\langle du, dv\rangle + \langle d^*u, d^*v\rangle = \langle f - P_{\mathfrak{H}} f, v\rangle,\ v \in V^k \cap V_k^* \cap \mathfrak{H}^{k\perp}.$$

- *Mixed weak formulation.* Find $\sigma \in V^{k-1}$, $u \in V^k$, and $p \in \mathfrak{H}^k$ s.t.

$$\begin{aligned} \langle \sigma, \tau\rangle - \langle u, d\tau\rangle &= 0, & \tau &\in V^{k-1}, \\ \langle d\sigma, v\rangle + \langle du, dv\rangle + \langle p, v\rangle &= \langle f, v\rangle, & v &\in V^k, \\ \langle u, q\rangle &= 0, & q &\in \mathfrak{H}^k. \end{aligned}$$

Equivalence and well-posedness

Theorem

Let $f \in W^k$. Then $u \in W^k$ solves the strong formulation $\iff$ it solves the primal weak formulation. Moreover, in this case, if we set $\sigma = d^* u$ and $p = P_{\mathfrak{H}} u$, then the triple (σ, u, p) solves the mixed weak formulation. Finally, if some (σ, u, p) solves the mixed weak formulation, then $\sigma = d^* u$, $p = \mathcal{P}_{\mathfrak{H}} u$, and u solves the strong and primal formulations of the problem.

Theorem

For each $f \in W^k$ there exists a unique solution. Moreover

$$\|u\| + \|du\| + \|d^* u\| + \|dd^* u\| + \|d^* du\| \leq c\|f - P_{\mathfrak{H}} f\|.$$

The constant depends only on the Poincaré inequality constant c^P.

Proof of well-posedness

We used the mixed formulation. Set

$$B(\sigma, u, p; \tau, v, q) = \langle \sigma, \tau \rangle - \langle u, d\tau \rangle - \langle d\sigma, v \rangle - \langle du, dv \rangle - \langle p, v \rangle - \langle u, q \rangle$$

We must prove the inf-sup condition: $\forall\ (\sigma, u, p)\ \exists\ (\tau, v, q)$ s.t.

$$B(\sigma, u, p; \tau, v, q) \geq \gamma(\|\sigma\|_V + \|u\|_V + \|p\|)(\|\tau\|_V + \|v\|_V + \|q\|),$$

with $\gamma = \gamma(c^P) > 0$. Via the Hodge decomposition,

$$u = u_{\mathfrak{B}} + u_{\mathfrak{H}} + u_{\mathfrak{B}*} = d\rho + u_{\mathfrak{H}} + u_{\mathfrak{B}*}$$

with $\rho \in \mathfrak{Z}^{\perp_V}$. Then take

$$\tau = \sigma - \frac{1}{(c^P)^2}\rho, \quad v = -u - d\sigma - p, \quad q = p - u_{\mathfrak{H}}.$$

Hodge Laplacian and Hodge decomposition

- $f = dd^*u + P_{\mathfrak{H}}f + d^*du$ is the Hodge decomposition of f
- Define $K : W^k \to D(L^k)$ by $Kf = u$ (bdd lin op).
- $P_{\mathfrak{B}} = dd^*K, \quad P_{\mathfrak{B}^*} = d^*dK$
- If $f \in V$, then $Kdf = dKf$.
- If $f \in \mathfrak{B}$, then $dKf = 0$. Since $Kf \perp \mathfrak{H}$, $Kf \in \mathfrak{B}$.
- $\mathfrak{B}$ *problem:* If $f \in \mathfrak{B}$, then $u = Kf$ solves

$$dd^*u = f, \quad du = 0, \quad u \perp \mathfrak{H}.$$

- $\mathfrak{B}^*$ *problem:* If $f \in \mathfrak{B}^*$, then $u = Kf$ solves

$$d^*du = f, \quad d^*u = 0, \quad u \perp \mathfrak{H}.$$

The Hodge Laplacian on a domain in 3D

$$0 \to H^1 \xrightarrow{\mathrm{grad}} H(\mathrm{curl}) \xrightarrow{\mathrm{curl}} H(\mathrm{div}) \xrightarrow{\mathrm{div}} L^2 \to 0$$

$$0 \leftarrow L^2 \xleftarrow{-div} \mathring{H}(\mathrm{div}) \xleftarrow{\mathrm{curl}} \mathring{H}(\mathrm{curl}) \xleftarrow{-\,\mathrm{grad}} \mathring{H}^1 \leftarrow 0$$

k	$L^k = d^*d + dd^*$	BCs imposed on...		$V^{k-1} \times V^k$
0	$-\Delta$		$\partial u/\partial n$	H^1
1	$\mathrm{curl}\,\mathrm{curl} - \mathrm{grad}\,\mathrm{div}$	$u \cdot n$	$\mathrm{curl}\, u \times n$	$H^1 \times H(\mathrm{curl})$
2	$-\,\mathrm{grad}\,\mathrm{div} + \mathrm{curl}\,\mathrm{curl}$	$u \times n$	$\mathrm{div}\, u$	$H(\mathrm{curl}) \times H(div)$
3	$-\Delta$	u		$H(\mathrm{div}) \times L^2$

essential BC for primal form.

natural BC for primal form.

Approximation of Hilbert complexes

Why mixed methods?

Naively, we might try to discretize the primal formulation with finite elements. This works in some circumstances, but we have seen two ways in which it can fail. It is not easy to construct a dense family of subspaces of the primal energy space $V^k \cap V_k^* \cap \mathfrak{H}^k$.

We therefore consider finite element discretizations of the mixed formulation:

Given $f \in W^k$, find $\sigma \in V^{k-1}$, $u \in V^k$, and $p \in \mathfrak{H}^k$ s.t.

$$\begin{aligned} \langle \sigma, \tau \rangle - \langle u, d\tau \rangle &= 0, & \tau &\in V^{k-1}, \\ \langle d\sigma, v \rangle + \langle du, dv \rangle + \langle p, v \rangle &= \langle f, v \rangle, & v &\in V^k, \\ \langle u, q \rangle &= 0, & q &\in \mathfrak{H}^k. \end{aligned}$$

Galerkin method

- Choose f.d. subspaces $V_h^j \subset V^j$
- $\mathfrak{Z}_h^j = \{v \in V_h^j | dv = 0\} \subset \mathfrak{Z}^j \qquad \mathfrak{B}_h^j = \{dv | v \in V_h^{j-1}\} \subset \mathfrak{B}^j$

$$\mathfrak{H}_h^j = \{v \in \mathfrak{Z}_h^j | v \perp \mathfrak{B}_h^j\}$$

Given $f \in W^k$, find $\sigma_h \in V_h^{k-1}$, $u_h \in V_h^k$, and $p_h \in \mathfrak{H}_h^k$ s.t.

$$\begin{aligned} \langle \sigma_h, \tau \rangle - \langle u_h, d\tau \rangle &= 0, & \tau &\in V_h^{k-1}, \\ \langle d\sigma_h, v \rangle + \langle du_h, dv \rangle + \langle p_h, v \rangle &= \langle f, v \rangle, & v &\in V_h^k, \\ \langle u_h, q \rangle &= 0, & q &\in \mathfrak{H}_h^k. \end{aligned}$$

If $\mathfrak{H}_h^k \not\subseteq \mathfrak{H}^k$ this is a *nonconforming method*.

For *any* choice of the V_h^j there exists a unique solution.
However, the consistency, stability, and accuracy of the discrete solution depends vitally on the choice of subspaces.

Key assumptions

We need the spaces $V_h^j \subset V^j$ (at least for $j = k-1, k, k+1$) to satisfy three properties:

1. Approximation property: Of course V_h^j must afford good approximation of elements of V^j. This can be formalized with respect to a family of subspaces parametrized by h by requiring

   $$\lim_{h \to 0} \inf_{v \in V_h^j} \|w - v\|_V = 0, \quad w \in V^j$$

   (or $= O(h^r)$ for w in some dense subspace, or ...)
2. Subcomplex property: $dV_h^{k-1} \subset V_h^k$ and $dV_h^k \subset V_h^{k+1}$, so

   $$\cdots \to V_h^{k-1} \xrightarrow{d} V_h^k \xrightarrow{d} V_h^{k+1} \to \cdots$$

   is a subcomplex.

Bounded cochain projection

3. Bounded cochain projection: Most important, we assume that there exists a *cochain map* from the H-complex to the subcomplex which is a *projection* and is *bounded*.

$$\begin{tikzcd}
V^{k-1} \arrow[r, "d"] \arrow[d, "\pi_h^{k-1}"'] & V^k \arrow[r, "d"] \arrow[d, "\pi_h^k"'] & V^{k+1} \arrow[d, "\pi_h^{k+1}"'] \\
V_h^{k-1} \arrow[r, "d"] & V_h^k \arrow[r, "d"] & V_h^{k+1}
\end{tikzcd}$$

- For now, boundedness is in V-norm: $\|\pi_h v\|_V \leq c\|v\|_V$. But later we will need W-boundedness, which is a stronger requirement.
- A bounded projection is *quasioptimal*:

$$\|v - \pi_h v\|_V \leq c \inf_{w \in V_h^j} \|v - w\|_V, \quad v \in V^j$$

First consequences from the assumptions

From the subcomplex property

$$\cdots \to V_h^{k-1} \xrightarrow{d} V_h^k \xrightarrow{d} V_h^{k+1} \cdots \to$$

is itself a closed H-complex. (We take $W_h^k = V_h^k$ but with the W-norm.)

Therefore there is a discrete adjoint operator d_h^* (its domain is all of W_h^k), a discrete Hodge decomposition

$$V_h^k = \mathfrak{B}_h^k \oplus \mathfrak{H}_h^k \oplus \mathfrak{B}_{kh}^*.$$

and a discrete Poincaré inequality

$$\|z\|_V \le c_h^P \|dz\|, \quad z \in \mathfrak{Z}_h^{k\perp_V}.$$

Preservation of cohomology

Theorem

Given: a closed H-complex, and a choice of f.d. subspaces satisfying the subcomplex property and admitting a V-bdd cochain projection π_h. Assume also the (very weak) approximation property

$$\|q - \pi_h q\| < \|q\|, \quad 0 \neq q \in \mathfrak{H}^k.$$

Then π_h induces an isomorphism from $\mathfrak{H}^k$ onto $\mathfrak{H}^k_h$. Moreover,

$$\operatorname{gap}\bigl(\mathfrak{H}, \mathfrak{H}_h\bigr) \le \sup_{\substack{q \in \mathfrak{H} \\ \|q\|=1}} \|q - \pi_h q\|_V.$$

$$\operatorname{gap}(\mathfrak{H}, \mathfrak{H}_h) := \max\left(\sup_{\substack{u \in \mathfrak{H} \\ \|u\|=1}} \inf_{v \in \mathfrak{H}_h} \|u - v\|_V, \sup_{\substack{v \in \mathfrak{H}_h \\ \|v\|=1}} \inf_{u \in \mathfrak{H}} \|u - v\|_V \right).$$

Uniform Poincaré inequality and stability

Theorem

Given: a closed H-complex, and a choice of f.d. subspaces satisfying the subcomplex property and admitting a V-bdd cochain projection π_h. Then

$$\|v\|_V \le c^P \|\pi_h\| \|dv\|_V, \quad v \in \mathfrak{Z}_h^{k\perp} \cap V_h^k.$$

Corollary (Stability and quasioptimality of the mixed method)

The mixed method is stable (uniform inf-sup condition) and satisfies

$$\begin{aligned}\|\sigma - \sigma_h\|_V + \|u - u_h\|_V + \|p - p_h\| \\ \le C\Big(\inf_{\tau \in V_h^{k-1}} \|\sigma - \tau\|_V + \inf_{v \in V_h^k} \|u - v\|_V + \inf_{q \in V_h^k} \|p - q\|_V \\ + \mu \inf_{v \in V_h^k} \|P_{\mathfrak{B}} u - v\|_V\Big),\end{aligned}$$

where $\mu = \mu_h = \sup_{r \in \mathfrak{H}^k, \|r\|=1} \|(I - \pi_h) r\|.$

Improved error estimates

In addition to $\mu = \|(I - \pi_h)P_{\mathfrak{H}}\|$, define $\delta, \eta = o(1)$ by

$$\delta = \|(I - \pi_h)K\|_{\mathrm{Lin}(W,W)}, \quad \eta = \|(I - \pi_h)d^{[*]}K\|_{\mathrm{Lin}(W,W)}.$$

When $V_h^k \supset \mathcal{P}_r, \quad \mu = O(h^{r+1}), \quad \eta = O(h), \quad \delta = \begin{cases} O(h^2), & r > 0, \\ O(h), & r = 0, \end{cases}$

Theorem

Given: an H-complex satisfying the compactness property, and a choice of f.d. subspaces satisfying the subcomplex property and admitting a W-bdd cochain projection π_h. Then

$$\|d(\sigma - \sigma_h)\| \le cE(d\sigma), \qquad \|\sigma - \sigma_h\| \le c[E(\sigma) + \eta E(d\sigma)],$$

$$\|d(u - u_h)\| \le c\{E(du) + \eta[E(d\sigma) + E(p)]\},$$

$$\|u - u_h\| \le c\{E(u) + \eta[E(du) + E(\sigma)] \\ +(\eta^2 + \delta)[E(d\sigma) + E(p)] + \mu E(P_{\mathfrak{B}}u)\}.$$

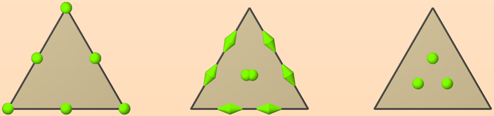
Numerical tests

$-\operatorname{grad}\operatorname{div} u + \operatorname{curl}\operatorname{rot} u = f$ in Ω (unit square), $u \cdot n = \operatorname{rot} u = 0$ on $\partial\Omega$
(magnetic BC)

$$0 \to H^1 \xrightarrow{\text{grad}} H(\text{rot}) \xrightarrow{\text{rot}} L^2 \to 0$$

$\sigma_h \in V_h^0 \subset H^1, \quad u_h \in V_h^1 \subset H(\text{rot})$

$$\begin{aligned}
\langle \sigma_h, \tau \rangle - \langle u_h, \operatorname{grad} \tau \rangle &= 0, && \tau \in V_h^{k-1}, \\
\langle \operatorname{grad} \sigma_h, v \rangle + \langle \operatorname{rot} u_h, \operatorname{rot} v \rangle + \langle p_h, v \rangle &= \langle f, v \rangle, && v \in V_h^k, \\
\langle u_h, q \rangle &= 0, && q \in \mathfrak{H}_h^k.
\end{aligned}$$

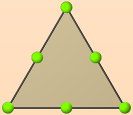


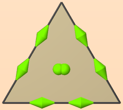
V_h^0 Lagrange $\quad$ V_h^1 R-T $\quad$ V_h^2 DG

All hypotheses are met. . .

Numerical solution of vector Laplacian, magnetic BC

$\|\sigma - \sigma_h\|$	rate	$\|\nabla(\sigma - \sigma_h)\|$	rate	$\|u - u_h\|$	rate	$\|\operatorname{rot}(u - u_h)\|$	rate
2.16e-04	3.03	2.63e-02	1.98	2.14e-03	1.99	1.17e-02	1.99
2.70e-05	3.00	6.60e-03	1.99	5.37e-04	1.99	2.93e-03	2.00
3.37e-06	3.00	1.65e-03	2.00	1.34e-04	2.00	7.33e-04	2.00
4.16e-07	3.02	4.14e-04	2.00	3.36e-05	2.00	1.83e-04	2.00
	3		2		2		2





Numerical solution of vector Laplacian, Dirichlet BC

For Dirichlet boundary conditions, $\sigma = -\operatorname{div} u$ is sought in H^1, u is sought in $\mathring{H}(\mathrm{rot})$ (the BC $u \cdot t = 0$ is essential, $u \cdot n = 0$ is natural).

There is no complex, so our theory does not apply.

$\|\sigma - \sigma_h\|$	rate	$\|\nabla(\sigma - \sigma_h)\|$	rate	$\|u - u_h\|$	rate	$\|\mathrm{rot}(u - u_h)\|$	rate
1.90e-02	1.62	2.53e+00	0.63	1.22e-03	2.01	1.55e-02	1.58
6.36e-03	1.58	1.68e+00	0.60	3.05e-04	2.00	5.33e-03	1.54
2.18e-03	1.54	1.14e+00	0.56	7.63e-05	2.00	1.85e-03	1.52
7.58e-04	1.52	7.89e-01	0.53	1.91e-05	2.00	6.49e-04	1.51
	1.5		0.5		2		1.5

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Eigenvalue problems

Find $\lambda \in \mathbb{R}$, $0 \neq u \in D(L)$ s.t. $Lu = \lambda u$, $u \perp \mathfrak{H}$

$$\lambda\|u\|^2 = \|du\|^2 + \|d^*u\|^2 > 0 \qquad \text{so} \qquad \lambda > 0 \text{ and } Ku = \lambda^{-1}u.$$

By the compactness property, $K : W^k \to W^k$ is compact and self-adjoint, so $0 < \lambda_1 \leq \lambda_2 \leq \cdots \to \infty$.
Denote by v_i corresponding orthonormal eigenvalues, $E_i = \mathbb{R}v_i$.

Mixed discretization:

Find $\lambda_h \in \mathbb{R}$, $0 \neq (\sigma_h, u_h, p_h) \in V_h^{k-1} \times V_h^k \times \mathfrak{H}_h^k$ s.t.

$$\begin{aligned} \langle \sigma_h, \tau \rangle - \langle u_h, d\tau \rangle &= 0, & \tau &\in V_h^{k-1}, \\ \langle d\sigma_h, v \rangle + \langle du_h, dv \rangle + \langle p_h, v \rangle &= \lambda_h \langle u_h, v \rangle, & v &\in V_h^k, \\ \langle u_h, q \rangle &= 0, & q &\in \mathfrak{H}_h^k. \end{aligned}$$

$0 < \lambda_{1h} \leq \lambda_{2h} \leq \ldots \leq \lambda_{N_h h}$, $\quad v_{ih}$ orthonormal, $\quad E_{ih} = \mathbb{R}v_{ih}$

Convergence of eigenvalue problems

Let $\sum_{i=1}^{m(j)} E_i$ be the span of the eigenspaces of the first j *distinct* eigenvalues. The method *converges* if $\forall\, j, \epsilon > 0$, $\exists\, h_0 > 0$ s.t.

$$\max_{1\le i\le m(j)} |\lambda_i - \lambda_{ih}| \le \epsilon \quad \text{and} \quad \operatorname{gap}\left(\sum_{i=1}^{m(j)} E_i, \sum_{i=1}^{m(j)} E_{i,h}\right) \le \epsilon \qquad \text{if } h \le h_0.$$

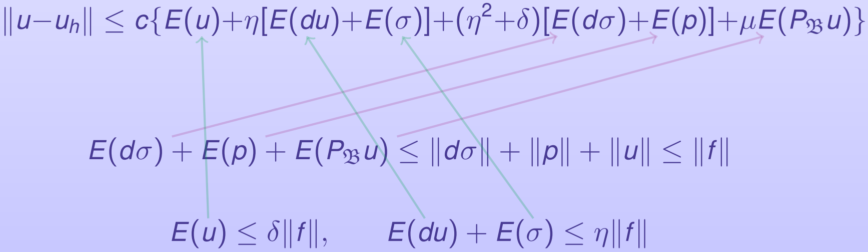
A sufficient (and necessary) condition for eigenvalue convergence is operator norm convergence of the discrete solution operator $K_h P_h$ to K (Kato, Babuska–Osborn, Boffi–Brezzi–Gastaldi):

$W \to W_h$ orthog.

The mixed discretization of the eigenvalue problem converges if

$$\lim_{h\to 0} \|K_h P_h - K\|_{\mathcal{L}(W,W)} = 0.$$

Eigenvalue convergence follows from improved estimates



$$\|u-u_h\| \le c\{E(u)+\eta[E(du)+E(\sigma)]+(\eta^2+\delta)[E(d\sigma)+E(p)]+\mu E(P_{\mathfrak{B}}u)\}$$

$$E(d\sigma)+E(p)+E(P_{\mathfrak{B}}u) \le \|d\sigma\|+\|p\|+\|u\| \le \|f\|$$

$$E(u) \le \delta\|f\|, \qquad E(du)+E(\sigma) \le \eta\|f\|$$

Therefore

$$\|(K-K_hP_h)f\| \le \delta+\eta^2+\mu \to 0$$

Rates of convergence also follow, included doubled convergence rates for eigenvalues…

Exterior calculus

The de Rham complex for a domain in $\mathbb{R}^n$

1-D: $0 \to C^\infty(\Omega) \xrightarrow{d/dx} C^\infty(\Omega) \to 0$

2-D: $0 \to C^\infty(\Omega) \xrightarrow{\text{grad}} C^\infty(\Omega, \mathbb{R}^2) \xrightarrow{\text{rot}} C^\infty(\Omega) \to 0$

3-D: $0 \to C^\infty(\Omega) \xrightarrow{\text{grad}} C^\infty(\Omega, \mathbb{R}^3) \xrightarrow{\text{curl}} C^\infty(\Omega, \mathbb{R}^3) \xrightarrow{\text{div}} C^\infty(\Omega) \to 0$

n-D: $0 \to \Lambda^0(\Omega) \xrightarrow{d} \Lambda^1(\Omega) \xrightarrow{d} \Lambda^2(\Omega) \xrightarrow{d} \cdots \xrightarrow{d} \Lambda^n(\Omega) \to 0$

The space $\Lambda^k(\Omega) = C^\infty(\Omega, \mathbb{R}^{n\times\cdots\times n}_{\text{skw}})$, the space of smooth *differential k-forms* on Ω.

- *Exterior derivative:* $d^k : \Lambda^k(\Omega) \to \Lambda^{k+1}(\Omega)$
- *Integral of a k-form over an oriented k-simplex:* $\int_f v \in \mathbb{R}$
- *Stokes theorem:* $\int_c du = \int_{\partial c} u, \quad u \in \Lambda^k, \; c \in C_k$
- All this works on *any smooth manifold*

Exterior algebra

Multilinear forms on an n-dimensional vector space V

- $\mathrm{Lin}^k\, V$: k-linear maps $\omega : \overbrace{V \times \cdots \times V}^{k \text{ times}} \to \mathbb{R}$ $\mathrm{Lin}^0\, V := \mathbb{R}$

  $\mathrm{Lin}^1\, V = V^*$ covectors, 1-forms
- tensor product: $(\omega \otimes \mu)(v_1, \ldots, v_{j+k}) = \omega(v_1, \ldots, v_j)\mu(v_{j+1}, \ldots, v_{j+k})$
- $\dim \mathrm{Lin}^k\, V = n^k$
- dual basis for $\mathrm{Lin}^1\, \mathbb{R}^n$: dx^1,…,dx^n with $dx^i(e_j) = \delta_{ij}$
- basis for $\mathrm{Lin}^k\, \mathbb{R}^n$: $dx^{\sigma_1} \otimes \cdots \otimes dx^{\sigma_k}$, $1 \le \sigma_1, \ldots, \sigma_k \le n$

Alternating multilinear forms (algebraic k-forms)

- $\omega \in \mathrm{Alt}^k V$ if $\omega(\ldots, v_i, \ldots, v_j, \ldots) = -\omega(\ldots, v_j, \ldots, v_i, \ldots)$
- $\dim \mathrm{Alt}^k V = \binom{n}{k}$
- *skew part:* $(\mathrm{skw}\, \omega)(v_1, \ldots, v_k) = \frac{1}{k!} \sum_{\sigma \in \Sigma_k} \mathrm{sign}(\sigma) \omega(v_{\sigma_1}, \ldots, v_{\sigma_k})$
- *exterior product:* $\omega \wedge \mu = \binom{j+k}{j} \mathrm{skw}(\omega \otimes \mu)$
- basis for $\mathrm{Alt}^k \mathbb{R}^n$: $dx^{\sigma_1} \wedge \cdots \wedge dx^{\sigma_k}$, $1 \le \sigma_1 < \ldots < \sigma_k \le n$

Exterior algebra continued

- If $\omega \in \mathrm{Alt}^k V$, $v \in V$, the *interior product* $\omega \lrcorner v \in \mathrm{Alt}^{k-1} V$ is

  $\omega \lrcorner v(v_1, \ldots, v_{k-1}) = \omega(v, v_1, \ldots, v_{k-1}) \quad (\omega \wedge \eta)\lrcorner v = (\omega \lrcorner v) \wedge \eta \pm \omega \wedge (\eta \lrcorner v)$
- If V has an *inner product*, pick any orthonormal basis $v_1, \ldots, v_n$ and define

  $\langle \omega, \eta \rangle = \sum_\sigma \omega(v_{\sigma(1)}, \ldots, v_{\sigma(k)})\eta(v_{\sigma(1)}, \ldots, v_{\sigma(k)}), \quad \omega, \eta \in \mathrm{Alt}^k V$

  (sum over $\sigma : \{1, \ldots, k\} \to \{1, \ldots, n\}$ increasing).
- $\dim \mathrm{Alt}^n V = 1$. Fix the *volume form* by $\mathrm{vol}(v_1, \ldots, v_n) = \pm 1$.
  An orientation for V fixes the sign.
- *Hodge star:* $\star : \mathrm{Alt}^k V \xrightarrow{\cong} \mathrm{Alt}^{n-k} V$ defined by

  $\omega \wedge \mu = \langle \star\omega, \mu \rangle \,\mathrm{vol}, \quad \omega \in \mathrm{Alt}^k V,\ \mu \in \mathrm{Alt}^{n-k} V \qquad \star\star\omega = \pm\omega$
- On $\mathbb{R}^n$, $\mathrm{vol} = dx^1 \wedge \cdots \wedge dx^n = \det$, $\quad \star\, dx^{\sigma_1} \wedge \cdots \wedge dx^{\sigma_k} = \pm dx^{\sigma^*_1} \wedge \cdots \wedge dx^{\sigma^*_{n-k}}$
- *Pullback:* If $L : V \to W$ linear, $L^* : \mathrm{Alt}^k W \to Alt^k V$ is defined

  $L^*\omega(w_1, \ldots, w_k) = \omega(Lw_1, \ldots, Lw_n)$

Exterior algebra in $\mathbb{R}^3$

vector proxy	$\mathrm{Alt}^0\mathbb{R}^3 = \mathbb{R}$ $\mathrm{Alt}^1\mathbb{R}^3 \xrightarrow{\cong} \mathbb{R}^3$ $\mathrm{Alt}^2\mathbb{R}^3 \xrightarrow{\cong} \mathbb{R}^3$ $\mathrm{Alt}^3\mathbb{R}^3 \xrightarrow{\cong} \mathbb{R}$	$c \leftrightarrow c$ $u_1\,dx_1 + u_2\,dx_2 + u_3\,dx_3 \leftrightarrow u$ $u_1\,dx_2\wedge dx_3 - u_2\,dx_1\wedge dx_3 + u_3\,dx_1\wedge dx_2 \leftrightarrow u$ $c \leftrightarrow c\,dx_1\wedge dx_2\wedge dx_3$
exterior prod.	$\wedge : \mathrm{Alt}^1\mathbb{R}^3 \times \mathrm{Alt}^1\mathbb{R}^3 \to \mathrm{Alt}^2\mathbb{R}^3$ $\wedge : \mathrm{Alt}^1\mathbb{R}^3 \times \mathrm{Alt}^2\mathbb{R}^3 \to \mathrm{Alt}^3\mathbb{R}^3$	$\times : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ $\cdot : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$
pullback	$L^* : \mathrm{Alt}^0\mathbb{R}^3 \to \mathrm{Alt}^0\mathbb{R}^3$ $L^* : \mathrm{Alt}^1\mathbb{R}^3 \to \mathrm{Alt}^1\mathbb{R}^3$ $L^* : \mathrm{Alt}^2\mathbb{R}^3 \to \mathrm{Alt}^2\mathbb{R}^3$ $L^* : \mathrm{Alt}^3\mathbb{R}^3 \to \mathrm{Alt}^3\mathbb{R}^3$	$\mathrm{id} : \mathbb{R} \to \mathbb{R}$ $L^T : \mathbb{R}^3 \to \mathbb{R}^3$ $\mathrm{adj}\,L : \mathbb{R}^3 \to \mathbb{R}^3$ $(\det L) : \mathbb{R} \to \mathbb{R} \quad (c \mapsto c \det L)$
interior prod.	$\lrcorner v : \mathrm{Alt}^1\mathbb{R}^3 \to \mathrm{Alt}^0\mathbb{R}^3$ $\lrcorner v : \mathrm{Alt}^2\mathbb{R}^3 \to \mathrm{Alt}^1\mathbb{R}^3$ $\lrcorner v : \mathrm{Alt}^3\mathbb{R}^3 \to \mathrm{Alt}^2\mathbb{R}^3$	$v\cdot : \mathbb{R}^3 \to \mathbb{R}$ $v\times : \mathbb{R}^3 \to \mathbb{R}^3$ $v : \mathbb{R} \to \mathbb{R}^3 \quad (c \mapsto cv)$
inner prod.	inner product on $\mathrm{Alt}^k\mathbb{R}^3$	dot product on $\mathbb{R}$ and $\mathbb{R}^3$
Hodge star	$\star : \mathrm{Alt}^0\mathbb{R}^3 \to \mathrm{Alt}^3\mathbb{R}^3$ $\star : \mathrm{Alt}^1\mathbb{R}^3 \to \mathrm{Alt}^2\mathbb{R}^3$	$\mathrm{id} : \mathbb{R} \to \mathbb{R}$ $\mathrm{id} : \mathbb{R}^3 \to \mathbb{R}^3$

Exterior calculus

- *A differential k-form* on a manifold M is a map $x \in M \mapsto \omega_x \in \mathrm{Alt}^k T_x M$.
  ω takes a point $x \in M$ and k-tangent vectors and returns a number.
  0-forms are functions, 1-forms are covector fields.
  We write $\omega \in \Lambda^k(M)$ or $C\Lambda^k(M)$ if its continuous, $C^\infty\Lambda^k(M)$ if its smooth, etc.
- If $M = \Omega \subset \mathbb{R}^n$, $\omega : \Omega \to \mathrm{Alt}^k\mathbb{R}^n$. The general element of $\Lambda^k(\Omega)$ is $\omega = \sum_\sigma a_\sigma\, dx^{\sigma_1}\wedge\cdots\wedge dx^{\sigma_k}$ with $a_\sigma : \Omega \to \mathbb{R}$.
- A smooth map $\phi : M \to M'$, induces a linear maps $\phi'_x : T_x M \to T_x M'$ and so a *pullback* $\phi^* : \Lambda^k(M') \to \Lambda^k(M)$ on differential forms:

  $(\phi^*\omega)_x = {\phi'_x}^*\omega_{\phi(x)}$ or $(\phi^*\omega)_x(v_1, \ldots, v_k) = \omega_{\phi(x)}(\phi'_x v_1, \ldots, \phi'_x v_k)$

  $\phi^*(\omega\wedge\mu) = (\phi^*\omega)\wedge(\phi^*\mu)$ pullback of inclusion defines trace
- For $\omega = a\, dx^{\sigma_1}\wedge\cdots\wedge dx^{\sigma_k} \in \Lambda^k(\Omega)$, the *exterior derivative*

  $$d\omega = \sum_\sigma \sum_{k=1}^n \frac{\partial a_\sigma}{\partial x^k}\, dx^k\wedge dx^{\sigma_1}\wedge\cdots\wedge dx^{\sigma_k} \in \Lambda^{k+1}(\Omega).$$ It satisfies

  $$d \circ d = 0, \quad d(\omega\wedge\mu) = (d\omega)\wedge\mu \pm \omega\wedge d\mu, \quad \phi^*(d\omega) = d(\phi^*\omega)$$
- We may use any coordinate chart to define $d : \Lambda^k(M) \to \Lambda^{k+1}(M)$.

Exterior calculus continued

- A differential n-form on an oriented n-dim'l manifold M may be *integrated* very geometrically (w/o requiring a metric or measure): $\int_M \omega \in \mathbb{R}, \quad \omega \in \Lambda^n(M)$

  $\int_M \phi^* \omega = \int_{M'} \omega, \omega \in \Lambda^n(M')$ if ϕ preserves orientation.

  For $\omega = f(x)\,\mathrm{vol}$ on $\Omega \subset \mathbb{R}^n$ we get what notation suggests.

- *Stokes theorem:* $\int_\Omega d\omega = \int_{\partial\Omega} \mathrm{tr}\,\omega, \quad \omega \in \Lambda^{k-1}(\Omega)$

  Combining with Leibniz, we get the *integration by parts* formula

  $$\int_\Omega d\omega\wedge\eta = \pm\int_\Omega \omega\wedge d\eta + \int_{\partial\Omega} \mathrm{tr}\,\omega\wedge \mathrm{tr}\,\eta, \quad \omega \in \Lambda^k(\Omega),\ \eta \in \Lambda^{n-k-1}(\Omega)$$

- For M an *oriented Riemannian manifold* we have inner prod and $\star$ on $\mathrm{Alt}^k\, T_x M$ and can define: $\langle \omega, \eta\rangle_{L^2\Lambda^k} = \int_\Omega \langle \omega_x, \eta_x\rangle\,\mathrm{vol} = \int \omega\wedge\star\,\eta$

- This allows us to rewrite the *integration by parts* formula

  $$\langle d\omega, \mu\rangle = \langle \omega, \delta\mu\rangle + \int_{\partial\Omega} \mathrm{tr}\,\omega\wedge \mathrm{tr}\star\mu, \quad \omega \in \Lambda^{k-1},\ \mu \in \Lambda^k,$$

  where $\delta\mu := \pm\star d\star\mu$ is the *coderivative* operator.

Vector proxies in $\mathbb{R}^n$

k	0	1	$n-1$	n
$\Lambda^k(\Omega)$	functions	vector fields	vector fields	functions
$\Lambda^k(\partial\Omega)$	functions	tang vctr flds	functions	0
$\operatorname{tr}: \Lambda^k(\Omega) \to \Lambda^k(\partial\Omega)$	$u\vert_{\partial\Omega}$	$\pi_t u\vert_{\partial\Omega}$	$u\vert_{\partial\Omega} \cdot n$	0
$d: \Lambda^k(\Omega) \to \Lambda^{k+1}(\Omega)$	grad	curl	div	0
$\delta: \Lambda^k(\Omega) \to \Lambda^{k-1}(\Omega)$	0	$-$ div	curl	$-$ grad
$\int_f: \Lambda^k(\Omega) \to \mathbb{R}$	$u(f)$	$\int_f u \cdot t \, d\mathcal{H}_1$	$\int_f u \cdot n \, d\mathcal{H}_{n-1}$	$\int_f u \, d\mathcal{H}_n$
$\phi^*: \Lambda^k(\Omega') \to \Lambda^k(\Omega)$	$u \circ \phi$	$(\phi'_x)^T (u \circ \phi)$	$(\operatorname{adj} \phi'_x)(u \circ \phi)$	$(\det \phi'_x)(u \circ \phi)$

$\dim f = k$

$\phi: \Omega \to \Omega'$

Piola transform

L^2 differential forms on a domain in $\mathbb{R}^n$

$\Omega \subset \mathbb{R}^n$ Lipschitz boundary

- $H\Lambda^k := \{u \in L^2\Lambda^k \,|\, du \in L^2\Lambda^{k+1}\}$

  $H^*\Lambda^k = \{u \in L^2\Lambda^k \,|\, \delta u \in L^2\Lambda^{k-1}\} = \star H\Lambda^{n-k}$
- $u \in H\Lambda^k(\Omega) \implies \operatorname{tr} u \in H^{-1/2}\Lambda^k(\partial\Omega)$

  $u \in H^*\Lambda^k(\Omega) \implies \operatorname{tr}\star u \in H^{-1/2}\Lambda^{n-k}(\partial\Omega)$
- $\mathring{H}\Lambda^k = \{u \in H\Lambda^k | \operatorname{tr} u = 0\}, \quad \mathring{H}^*\Lambda^k = \{u \in H^*\Lambda^k | \operatorname{tr}\star u = 0\}$

Theorem

If we view d as an unbdd operator $L^2\Lambda^k \to L^2\Lambda^{k+1}$ with domain $H\Lambda^k$, then $d^* = \delta$ with domain $\mathring{H}^*\Lambda^k$. Consequently

$$\mathfrak{H}^k = \{\omega \in L^2\Lambda^k \,|\, d\omega = 0,\ \delta\omega = 0,\ \operatorname{tr}\star\omega = 0\}.$$

L^2 de Rham complex on a domain in $\mathbb{R}^n$

L^2 de Rham complex:

$$0 \to H\Lambda^0 \xrightarrow{d} H\Lambda^1 \xrightarrow{d} \cdots \xrightarrow{d} H\Lambda^n \to 0$$

Dual complex:

$$0 \leftarrow \mathring{H}^*\Lambda^0 \xleftarrow{\delta} \mathring{H}^*\Lambda^1 \xleftarrow{\delta} \cdots \xleftarrow{\delta} \mathring{H}^*\Lambda^n \leftarrow 0$$

Theorem (R. Picard ’84)

For a domain (or Riemannian manifold) w/ Lipschitz boundary the compactness property holds: $H\Lambda^k \cap \mathring{H}^*\Lambda^k$ *is compact in* $L^2\Lambda^k$*.*

compactness property $\implies$ Fredholm $\implies$ closed

Finite element spaces of differential forms

Finite element spaces

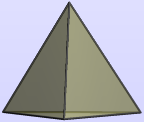
Goal: define *finite element spaces* $\Lambda_h^k \subset H\Lambda^k(\Omega)$ satisfying the hypotheses of approximation, subcomplexes, and bounded cochain projections.

A FE space is constructed by assembling three ingredients: Ciarlet '78

- A *triangulation* $\mathcal{T}$ consisting of polyhedral elements T
- For each T, a space of *shape functions* $V(T)$, typically polynomial
- For each T, a set of *DOFs*: a set of functionals on $V(T)$, each associated to a face of T. These must be *unisolvent*, i.e., form a basis for $V(T)^*$.

The FE space V_h is *defined* as functions piecewise in $V(T)$ with DOFs single-valued on faces. The DOFs determine (1) the interelement continuity, and (2) a projection operator into V_h.

Example: $H\Lambda^0 = H^1$: the Lagrange finite element family

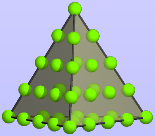


Elements $T \in \mathcal{T}_h$ are simplices in $\mathbb{R}^n$.

Shape fns: $V(T) = \mathcal{P}_r(T)$, some $r \geq 1$.

DOFs: $u \mapsto \int_f (\operatorname{tr}_f u) q,\ q \in \mathcal{P}_{r-d-1}(f),\ f \in \Delta(T),\quad d = \dim f$

- $v \in \Delta_0(T)$: $u \mapsto u(v)$
- $e \in \Delta_1(T)$: $u \mapsto \int_e (\operatorname{tr}_e u) q,\ q \in \mathcal{P}_{r-2}(e)$
- $f \in \Delta_2(T)$: $u \mapsto \int_f (\operatorname{tr}_f u) q,\ q \in \mathcal{P}_{r-3}(f)$
- T: $u \mapsto \int_T uq,\ q \in \mathcal{P}_{r-4}(T)$



Theorem

The number of DOFs $= \dim \mathcal{P}_r(T)$ *and they are unisolvent. The imposed continuity exactly forces inclusion in* H^1.

Unisolvence for Lagrange elements in n dimensions

Shape fns: $V(T) = \mathcal{P}_r(T)$, DOFs: $u \mapsto \int_f (\mathrm{tr}_f\, u) q,\ q \in \mathcal{P}_{r-d-1}(f),\ d = \dim f$

DOF count:

$$\#\mathrm{DOF} = \sum_{d=0}^{n} \overset{\substack{\#\Delta_d(T)\\ \searrow}}{\binom{n+1}{d+1}} \overset{\substack{\dim \mathcal{P}_{r-d-1}(f_d)\\ \downarrow}}{\binom{r-1}{d}} = \overset{\substack{\dim \mathcal{P}_r(T)\\ \swarrow}}{\binom{r+n}{n}} = \dim \mathcal{P}_r(T).$$

Unisolvence proved by induction on dimension ($n = 1$ is obvious).
Suppose $u \in \mathcal{P}_r(T)$ and all DOFs vanish. Let f be a facet of T. Note

- $\mathrm{tr}_f\, u \in \mathcal{P}_r(f)$
- the DOFs associated to f and its subfaces applied to u coincide with the Lagrange DOFs in $\mathcal{P}_r(f)$ applied to $\mathrm{tr}_f\, u$

Therefore $\mathrm{tr}_f\, u$ vanishes by the inductive hypothesis. Thus $u = (\prod_{i=0}^{n} \lambda_i) p, \quad p \in \mathcal{P}_{r-n-1}(T)$. Choose $q = p$ in the interior DOFs to see that $p = 0$.

Polynomial differential forms

- Polynomial diff. forms: $\mathcal{P}_r\Lambda^k(\Omega)$ $\sum_\sigma a_\sigma\, dx^{\sigma_1}\wedge\cdots\wedge dx^{\sigma_k},\ a_\sigma\in\mathcal{P}_r(\Omega)$

  Homogeneous polynomial diff. forms: $\mathcal{H}_r\Lambda^k(\Omega)$

- $\dim\mathcal{P}_r\Lambda^k = \binom{r+n}{r}\binom{n}{k} = \binom{r+n}{r+k}\binom{r+k}{k}$

  $\dim\mathcal{H}_r\Lambda^k = \binom{r+n-1}{r}\binom{n}{k} = \frac{n}{n+r}\binom{r+n}{r+k}\binom{r+k}{k}$

- *(Homogeneous) polynomial de Rham subcomplex:*

$$0 \to \mathcal{P}_r\Lambda^0 \xrightarrow{d} \mathcal{P}_{r-1}\Lambda^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{P}_{r-n}\Lambda^n \to 0$$

$$0 \to \mathcal{H}_r\Lambda^0 \xrightarrow{d} \mathcal{H}_{r-1}\Lambda^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{H}_{r-n}\Lambda^n \to 0$$

The Koszul complex

For $x \in \Omega \subset \mathbb{R}^n$, $T_x\Omega$ may be identified with $\mathbb{R}^n$, so the identity map can be viewed as a vector field.

- The Koszul differential $\kappa : \Lambda^k \to \Lambda^{k-1}$ is the contraction with the identity: $\kappa\omega = \omega \lrcorner \, \mathrm{id}$. Applied to polynomials it increases degree.
- $\kappa \circ \kappa = 0$ giving the *Koszul complex:*

$$0 \to \mathcal{P}_r\Lambda^n \xrightarrow{\kappa} \mathcal{P}_{r+1}\Lambda^{n-1} \xrightarrow{\kappa} \cdots \mathcal{P}_{r+n}\Lambda^0 \to 0$$

- $\kappa \, dx^i = x^i, \qquad \kappa(\omega \wedge \mu) = (\kappa\omega) \wedge \mu \pm \omega \wedge (\kappa\mu)$
- $\kappa(f \, dx^{\sigma_1} {}_\wedge \cdots {}_\wedge dx^{\sigma^k}) = f \sum_{i=1}^k (-)^i dx^{\sigma_1} {}_\wedge \cdots \widehat{dx^{\sigma_i}} \cdots {}_\wedge dx^{\sigma^k}$
- 3D Koszul complex:

$$0 \to \mathcal{P}_r\Lambda^3 \xrightarrow{x} \mathcal{P}_{r+1}\Lambda^2 \xrightarrow{\times x} \mathcal{P}_{r+2}\Lambda^1 \xrightarrow{\cdot x} \mathcal{P}_{r+3}\Lambda^0 \to 0$$

Theorem (Homotopy formula)

$$(d\kappa + \kappa d)\omega = (r+k)\omega, \quad \omega \in \mathcal{H}_r\Lambda^k.$$

Proof of the homotopy formula

$$(d\kappa + \kappa d)\omega = (r+k)\omega, \quad \omega \in \mathcal{H}_r\Lambda^k, \quad \omega \in \mathcal{H}_r\Lambda^k$$

Proof by induction on k. $k = 0$ is Euler's identity.
Assume true for $\omega \in \mathcal{H}_r\Lambda^{k-1}$, and verify it for $\omega \wedge dx^i$.

$$\begin{aligned} d\kappa(\omega \wedge dx^i) &= d(\kappa\omega \wedge dx^i + (-1)^{k-1}\omega \wedge x^i) \\ &= d(\kappa\omega) \wedge dx^i + (-1)^{k-1}(d\omega) \wedge x^i + \omega \wedge dx^i. \end{aligned}$$

$$\kappa d(\omega \wedge dx^i) = \kappa(d\omega \wedge dx^i) = \kappa(d\omega) \wedge dx^i + (-1)^k d\omega \wedge x^i.$$

$$(d\kappa + \kappa d)(\omega \wedge dx^i) = [(d\kappa + \kappa d)\omega] \wedge dx^i + \omega \wedge dx^i = (r+k)(\omega \wedge dx^i).$$

Consequences of the homotopy formula

- The polynomial de Rham complex is exact (except for constant 0-forms in the kernel). The Koszul complex is exact (except for constant 0-forms in the coimage).
- $\kappa d\omega = 0 \implies d\omega = 0, \quad d\kappa\omega = 0 \implies \kappa\omega = 0$
- $\mathcal{H}_r\Lambda^k = \kappa\mathcal{H}_{r-1}\Lambda^{k+1} \oplus d\mathcal{H}_{r+1}\Lambda^{k-1}$
- Define $\mathcal{P}_r^-\Lambda^k = \mathcal{P}_{r-1}\Lambda^k + \kappa\mathcal{H}_{r-1}\Lambda^{k+1}$
- $\mathcal{P}_r^-\Lambda^0 = \mathcal{P}_r\Lambda^0, \quad \mathcal{P}_r^-\Lambda^n = \mathcal{P}_{r-1}\Lambda^n, \quad$ else $\mathcal{P}_{r-1}\Lambda^k \subsetneq \mathcal{P}_r^-\Lambda^k \subsetneq \mathcal{P}_r\Lambda^k$
- $\dim \mathcal{P}_r^-\Lambda^k = \binom{r+n}{r+k}\binom{r+k-1}{k} = \frac{r}{r+k}\dim \mathcal{P}_r\Lambda^k$
- $\mathcal{R}(d|\mathcal{P}_r^-\Lambda^k) = \mathcal{R}(d|\mathcal{P}_r\Lambda^k), \quad \mathcal{N}(d|\mathcal{P}_r^-\Lambda^k) = \mathcal{N}(d|\mathcal{P}_{r-1}\Lambda^k)$
- The complex (with constant r)

$$0 \to \mathcal{P}_r^-\Lambda^0 \xrightarrow{d} \mathcal{P}_r^-\Lambda^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{P}_r^-\Lambda^n \to 0$$

  is exact (except for constant 0-forms).

Complexes mixing $\mathcal{P}_r$ and $\mathcal{P}_r^-$

On an n-D domain there are 2^{n-1} complexes beginning with $\mathcal{P}_r\Lambda^0$ (or ending with $\mathcal{P}_r\Lambda^n$). At each step we have two choices:

$$\begin{tikzcd}[row sep=small]
 & \mathcal{P}_r^-\Lambda^k & & \mathcal{P}_r^-\Lambda^k \\
\mathcal{P}_r\Lambda^{k-1} \arrow[ur] \arrow[dr] & \text{or} & \mathcal{P}_r^-\Lambda^{k-1} \arrow[ur] \arrow[dr] & \\
 & \mathcal{P}_{r-1}\Lambda^k & & \mathcal{P}_{r-1}\Lambda^k
\end{tikzcd}$$

In 3-D:

$$\begin{aligned}
&0 \to \mathcal{P}_r\Lambda^0 \xrightarrow{d} \mathcal{P}_r^-\Lambda^1 \xrightarrow{d} \mathcal{P}_r^-\Lambda^2 \xrightarrow{d} \mathcal{P}_{r-1}\Lambda^3 \to 0.\\
&0 \to \mathcal{P}_r\Lambda^0 \xrightarrow{d} \mathcal{P}_r^-\Lambda^1 \xrightarrow{d} \mathcal{P}_{r-1}\Lambda^2 \xrightarrow{d} \mathcal{P}_{r-2}\Lambda^3 \to 0,\\
&0 \to \mathcal{P}_r\Lambda^0 \xrightarrow{d} \mathcal{P}_{r-1}\Lambda^1 \xrightarrow{d} \mathcal{P}_{r-1}^-\Lambda^2 \xrightarrow{d} \mathcal{P}_{r-2}\Lambda^3 \to 0,\\
&0 \to \mathcal{P}_r\Lambda^0 \xrightarrow{d} \mathcal{P}_{r-1}\Lambda^1 \xrightarrow{d} \mathcal{P}_{r-2}\Lambda^2 \xrightarrow{d} \mathcal{P}_{r-3}\Lambda^3 \to 0,
\end{aligned}$$

The $P_r^- \Lambda^k$ family of simplicial FE differential forms

Given: a mesh $\mathcal{T}_h$ of simplices T, $r \geq 1$, $0 \leq k \leq n$, we define $\mathcal{P}_r^- \Lambda^k(\mathcal{T}_h)$ via:

Shape fns: $\mathcal{P}_r^- \Lambda^k(T)$

DOFs:

$$u \mapsto \int_f (\mathrm{tr}_f\, u) \wedge q, \quad q \in \mathcal{P}_{r+k-d-1} \Lambda^{d-k}(f),\ f \in \Delta(T), \quad d = \dim f \geq k$$

Theorem

The number of DOFs $= \dim \mathcal{P}_r^- \Lambda^k(T)$ *and they are unisolvent. The imposed continuity exactly enforces inclusion in* $H\Lambda^k$.

Dimension count

$$\begin{aligned}
\#\text{DOFs} &= \sum_{d\geq k} \#\Delta_d(T)\dim\mathcal{P}_{r+k-d-1}\Lambda^k(\mathbb{R}^d) \\
&= \sum_{d\geq k} \binom{n+1}{d+1}\binom{r+k-1}{d}\binom{d}{k} \\
&= \sum_{j\geq 0} \binom{n+1}{j+k+1}\binom{r+k-1}{j+k}\binom{j+k}{j}
\end{aligned}$$

Simplify using the identities

$$\binom{a}{b}\binom{b}{c} = \binom{a}{c}\binom{a-c}{a-b} \qquad \sum_{j\geq 0}\binom{a}{b+j}\binom{c}{j} = \binom{a+c}{a-b}$$

to get

$$\#\text{DOFs} = \binom{r+n}{r+k}\binom{r+k-1}{k} = \dim\mathcal{P}_r^-\Lambda^k$$

Proof of unisolvence for $\mathcal{P}_r^-\Lambda^k$

Lemma

If $u \in \mathring{\mathcal{P}}_{r-1}\Lambda^k(T)$ and $\int_T u\wedge q = 0\ \forall q \in \mathcal{P}_{r+k-n-1}\Lambda^{n-k}(T)$, then $u = 0$.

Proof: This can be proved by an explicit choice of test function. ☐

Proof of unisolvence: Suppose $u \in \mathcal{P}_r^-\Lambda^k(T)$ and all the DOFS vanish: $\int_f (\operatorname{tr}_f u)\wedge q = 0, \quad q \in \mathcal{P}_{r+k-d-1}\Lambda^{d-k}(f),\ f \in \Delta(T)$.
Then $\operatorname{tr}_f u \in \mathcal{P}_r^-\Lambda^k(f)$ and all its DOFs vanish. By induction on dimension, $\operatorname{tr} u$ vanishes on the boundary. So we need to show:

$$u \in \mathring{\mathcal{P}}_r^-\Lambda^k(T), \quad \int_T u\wedge q = 0\ \forall q \in \mathcal{P}_{r+k-n-1}\Lambda^{n-k}(T) \implies u = 0$$

In view of lemma, we just need to show $u \in \mathcal{P}_{r-1}\Lambda^k(T)$.

- By the homotopy formula, $u \in \mathcal{P}_r^-\Lambda^k$, $du = 0 \implies u \in \mathcal{P}_{r-1}\Lambda^k$. So it remains to show that $du = 0$.
- $du \in \mathring{\mathcal{P}}_{r-1}\Lambda^{k+1}(T)$,
  $\int_T du\wedge p = \pm\int_T u\wedge dp = 0\ \forall p \in \mathcal{P}_{r+k-n}\Lambda^{n-k-1}(T)$.
  Therefore $du = 0$ by the lemma (with $k \to k+1$).

The $P_r\Lambda^k$ family of simplicial FE differential forms

Given: a mesh $\mathcal{T}_h$ of simplices T, $r \geq 1$, $0 \leq k \leq n$, we define $\mathcal{P}_r\Lambda^k(\mathcal{T}_h)$ via:

Shape fns: $\mathcal{P}_r\Lambda^k(T)$

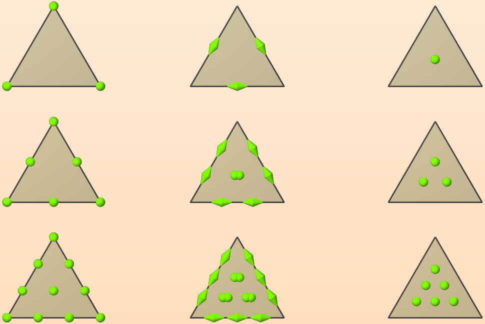
DOFs:

$$u \mapsto \int_f (\mathrm{tr}_f\, u)\wedge q, \quad q \in \mathcal{P}^-_{r+k-d}\Lambda^{d-k}(f),\ f \in \Delta(T), \quad d = \dim f \geq k$$

Theorem

The number of DOFs $= \dim \mathcal{P}_r\Lambda^k(T)$ *and they are unisolvent. The imposed continuity exactly enforces inclusion in* $H\Lambda^k$*.*

The $\mathcal{P}_r^-$ family in 2D



$\mathcal{P}_r^-\Lambda^0$ Lagrange

$\mathcal{P}_r^-\Lambda^1$ Raviart-Thomas '76

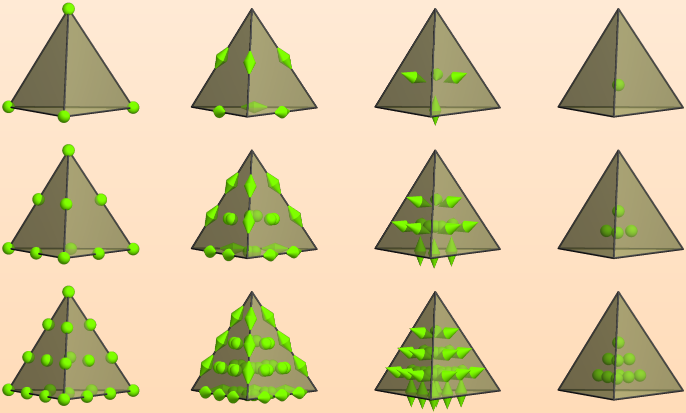
$\mathcal{P}_r^-\Lambda^2$ DG

$r = 1$

$r = 2$

$r = 3$

The $\mathcal{P}_r^-\Lambda^k$ family in 3D



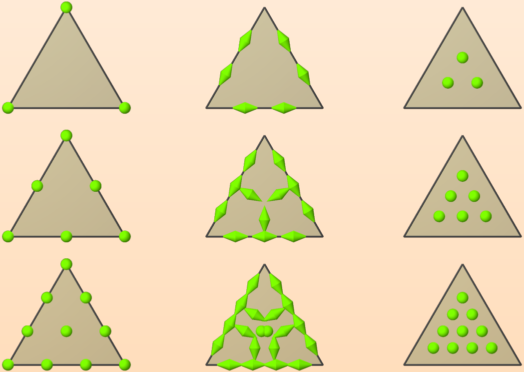
	$\mathcal{P}_r^-\Lambda^0$	$\mathcal{P}_r^-\Lambda^1$	$\mathcal{P}_r^-\Lambda^2$	$\mathcal{P}_r^-\Lambda^3$
	Lagrange	Nédélec ’80	Nédélec ’80	DG

$r = 1$

$r = 2$

$r = 3$

The $\mathcal{P}_r\Lambda^k$ family in 2D



$\mathcal{P}_r\Lambda^0$
Lagrange

$\mathcal{P}_r\Lambda^1$
Brezzi-Douglas-Marini ’85

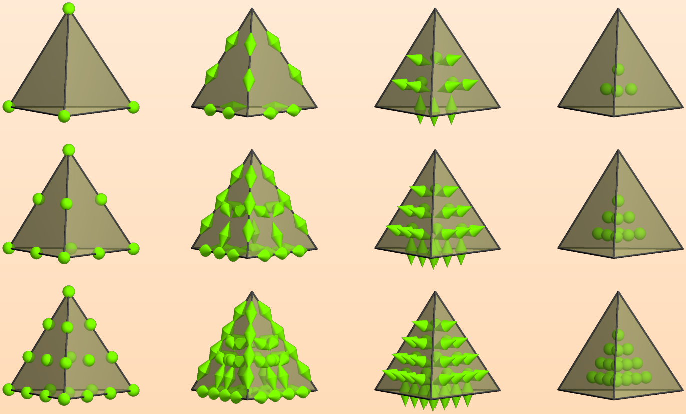
$\mathcal{P}_r\Lambda^2$
DG

$r = 1$

$r = 2$

$r = 3$

The $\mathcal{P}_r\Lambda^k$ family in 3D



	$\mathcal{P}_r\Lambda^0$	$\mathcal{P}_r\Lambda^1$	$\mathcal{P}_r\Lambda^2$	$\mathcal{P}_r\Lambda^3$
	Lagrange	Nédélec '86	Nédélec '86	DG
$r = 1$				
$r = 2$				
$r = 3$				

Application of the $\mathcal{P}_r$ and $\mathcal{P}_r^-$ families to the Hodge Laplacian

- The shape function spaces $\mathcal{P}_r\Lambda^k(T)$ and $\mathcal{P}_r^-\Lambda^k(T)$ combine into de Rham subcomplexes.
- The DOFs connect these spaces across elements to create subspaces of $H\Lambda^k(\Omega)$.

Therefore the assembled finite element spaces $\mathcal{P}_r\Lambda^k(\mathcal{T}_h)$ and $\mathcal{P}_r^-\Lambda^k(\mathcal{T}_h)$ combine into de Rham subcomplexes (in 2^{n-1} ways).

The DOFs of freedom determine projections from $\Lambda^k(\Omega)$ into the finite element spaces. From Stokes thm, these commute with d. Suitably modified, we obtain *bounded* cochain projections. Thus the abstract theory applies. We may use any two adjacent spaces in any of the complexes.

$$\left\{\begin{matrix} \mathcal{P}_r\Lambda^{k-1}(\mathcal{T}) \\ \text{or} \\ \mathcal{P}_r^-\Lambda^{k-1}(\mathcal{T}) \end{matrix}\right\} \xrightarrow{d} \left\{\begin{matrix} \mathcal{P}_r^-\Lambda^{k}(\mathcal{T}) \\ \text{or} \\ \mathcal{P}_{r-1}\Lambda^{k}(\mathcal{T}) \end{matrix}\right\}$$

Rates of convergence

Rates of convergence are determined by the improved error estimates from the abstract theory. They depend on

- The smoothness of the data f.
- The amount of elliptic regularity.
- The degree of of complete polynomials contained in the finite element spaces.

The theory delivers the best possible results: with sufficiently smooth data and elliptic regularity, the rate of convergence for each of the quantities u, du, σ, $d\sigma$, and p in the L^2 norm is the best possible given the degree of polynomials used for that quantity.

Eigenvalues converge as $O(h^{2r})$.

Historical notes

- The $\mathcal{P}_1^- \Lambda^k$ complex is in Whitney ’57 (Bossavit ’88).
- In ’76, Dodziuk and Patodi defined a finite difference approximation based on the Whitney forms to compute the eigenvalues of the Hodge Laplacian, and proved convergence. In retrospect, that method can be better viewed as a mixed finite element method. This was a step on the way to proving the Ray-Singer conjecture, completed in ’78 by W. Miller.
- The $\mathcal{P}_r \Lambda^k$ complex is in Sullivan ’78.
- Hiptmair gave a uniform treatment of the $\mathcal{P}_r^- \Lambda^k$ spaces in ’99.
- The unified treatment and use of the Koszul complex is in DNA-Falk-Winther ’06.

Bounded cochain projections

The DOFs defining $\mathcal{P}_r\Lambda^k(\mathcal{T}_h)$ and $\mathcal{P}_r^-\Lambda^k(\mathcal{T}_h)$ determine canonical projection operators Π_h from piecewise smooth forms in $H\Lambda^k$ onto Λ_h^k. However, Π_h is *not bounded* on $H\Lambda^k$ (much less uniformly bounded wrt h). Π_h *is* bounded on $C\Lambda^k$.

If we have a smoothing operator $R_{\epsilon,h} \in \mathrm{Lin}(L^2\Lambda^k, C\Lambda^k)$ such that $R_{\epsilon,h}$ commutes with d, we can define $Q_{\epsilon,h} = \Pi_h R_{\epsilon,h}$ and obtain a bounded operator $L^2\Lambda^k \to \Lambda_h^k$ which commutes with d (as suggested by Christiansen).

However Q_h will not be a projection. We correct this by using Schöberl's trick: if the finite dimensional operator

$$Q_{\epsilon,h}|_{\Lambda_h^k} : \Lambda_h^k \to \Lambda_h^k$$

is invertible, then

$$\pi_h := (Q_{\epsilon,h}|_{\Lambda_h^k})^{-1} Q_{\epsilon,h},$$

is a bounded commuting projection. It remains to get uniform bds on π_h.

The two key estimates

For this we need two key estimates for $Q_{\epsilon,h} := \Pi_h R_{\epsilon,h}$:

- For fixed ϵ, $Q_{\epsilon,h}$ is uniformly bounded:
  $\forall \epsilon > 0$ suff. small $\exists$ $c(\epsilon) > 0$ s.t.
  $$\sup_h \|Q_{\epsilon,h}\|_{\mathrm{Lin}(L^2,L^2)} \leq c(\epsilon)$$
- $\lim_{\epsilon \to 0} \|I - Q_{\epsilon,h}\|_{\mathrm{Lin}(L^2,L^2)} = 0$ uniformly in h

Theorem

Suppose that these two estimates hold and define $\pi_h := (Q_{\epsilon,h}|_{\Lambda_h^k})^{-1} Q_{\epsilon,h}$, where Λ_h^k is either $\mathcal{P}_r\Lambda^k(\mathcal{T}_h)$ or $\mathcal{P}_{r+1}^{-}\Lambda^k(T_h)$. Then, for h sufficiently small, π_h is a cochain projection onto Λ_h^k and

$$\|\omega - \pi_h\omega\| \leq ch^s\|\omega\|_{H^s\Lambda^k}, \quad \omega \in H^s\Lambda^k, \quad 0 \leq s \leq r+1.$$

The smoothing operator

The simplest definition is to take $R_{\epsilon,h}u$ to be an average over $y \in B_1$ of $(F^y_{\epsilon,h})^* u$ where $F^y_{\epsilon,h}(x) = x + \epsilon hy$:

$$R_{\epsilon,h}u(x) = \int_{B_1} \rho(y)[(F^y_{eh})^* u](x)\, dy$$

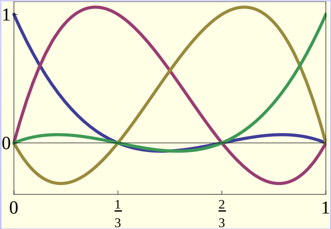
Needs modification near the boundary and for non-quasiuniform meshes.

The key estimates can be proven using macroelements and scaling.

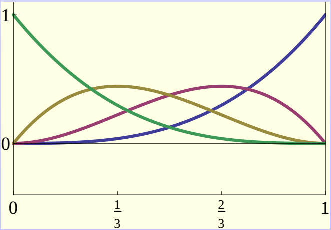
Bases for the spaces

Bases

Since the DOFs determine a basis for the dual space of a FE space, there is a corresponding basis for the FE space. An alternative is to use the Bernstein basis fns which are given explicitly in terms of the barycentric coordinates λ_i:



Basis of $\mathcal{P}_3$ dual to the nodal DOFs.



Bernstein basis functions $\lambda_0^i \lambda_1^j$.

For the $\mathcal{P}_r^- \Lambda^k$ and $\mathcal{P}_r \Lambda^k$ families in n-dimensions, there is of course again the basis determined by the DOFs. In addition, there is an explicit basis analogous to the Bernstein basis (DNA-Falk-Winter 2009).

Some computations with barycentric coordinates

- The barycentric coordinates $\lambda_0, \dots, \lambda_n$ form the dual basis for $\mathcal{P}_1(T) = \mathcal{P}_1^- \Lambda^0(T)$

- $d\lambda^1 \wedge \cdots \wedge d\lambda^n = c \,\mathrm{vol} \in \mathrm{Alt}^n T$ with

$$c = \frac{(d\lambda^1 \wedge \cdots \wedge d\lambda^n)(x_1 - x_0, \dots, x_n - x_0)}{\mathrm{vol}(x_1 - x_0, \dots, x_n - x_0)} = \frac{1}{n!|T|}$$

- More generally, $d\lambda_0 \wedge \dots \wedge \widehat{d\lambda_i} \wedge \dots \wedge d\lambda_n = \dfrac{(-1)^i}{n!|T|}\mathrm{vol}.$

- $\kappa d\lambda_i = \lambda_i - \lambda_i(0)$, so

$$\kappa(d\lambda_{\sigma_0} \wedge \cdots \wedge d\lambda_{\sigma_k}) = \sum_{i=0}^{k} (-1)^i \lambda_{\sigma_i} \, d\lambda_{\sigma_0} \wedge \dots \wedge \widehat{d\lambda_{\sigma_i}} \wedge \dots \wedge d\lambda_{\sigma_k} + \psi,$$

$\psi \in \mathcal{P}_0 \Lambda^k$.

The Whitney forms

- Define the *Whitney form* associated to the k-face f with vertices $x_{\sigma_0}, \dots, x_{\sigma_k}$ by

$$\phi_f = \sum_{i=0}^{k} (-1)^i \lambda_{\sigma_i}\, d\lambda_{\sigma_0} \wedge \dots \wedge \widehat{d\lambda_{\sigma_i}} \wedge \dots \wedge d\lambda_{\sigma_k} \in \mathcal{P}_1^- \Lambda^k$$

vertices: λ_i

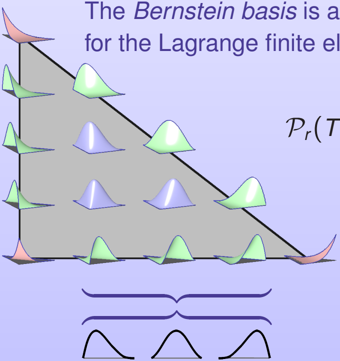
edges: $\lambda_i d\lambda_j - \lambda_j d\lambda_i$

triangles: $\lambda_i\, d\lambda_j \wedge d\lambda_k - \lambda_j\, d\lambda_i \wedge d\lambda_k + \lambda_k\, d\lambda_i \wedge d\lambda_j$

etc.

- If $f, g \in \Delta_k(T)$ then $\displaystyle\int_g \operatorname{tr}_g \phi_f = \begin{cases} 0, & g \neq f, \\ 1/k!, & g = f \end{cases}$
- $\therefore$ after normalization, the Whitney forms are a basis for $\mathcal{P}_1^- \Lambda^k$ dual to the DOFs.

Explicit geometric bases



The *Bernstein basis* is an explicit alternative to the Lagrange basis for the Lagrange finite elts.

$$\mathcal{P}_r = \operatorname{span}\{\, \lambda^\alpha := \lambda_0^{\alpha_0} \cdots \lambda_n^{\alpha_n} \mid |\alpha| = r \,\}$$

$$\mathcal{P}_r(T,f) := \operatorname{span}\{\, \lambda^\alpha \mid \operatorname{supp}\alpha = \{\sigma_0,\ldots,\sigma_k\},\ |\alpha| = r \,\}$$

$$\mathcal{P}_r(T) = \bigoplus_f \mathcal{P}_r(T,f)$$

$$\mathcal{P}_r(T,f) \xrightarrow[\text{tr}]{\cong} \mathring{\mathcal{P}}_r(f) \cong \mathcal{P}_{r-\dim f-1}(f)$$

There are similar geometric bases for all k:

$$\mathcal{P}_r\Lambda^k(T) = \bigoplus_{\dim f\ge k} \mathcal{P}_r\Lambda^k(T,f),\ \mathcal{P}_r\Lambda^k(T,f) \xrightarrow[\text{tr}]{\cong} \mathring{\mathcal{P}}_r\Lambda^k(f) \cong \mathcal{P}^-_{r+k-\dim f}\Lambda^{\dim f-k}(f)$$

$$\mathcal{P}^-_r\Lambda^k(T) = \bigoplus_{\dim f\ge k} \mathcal{P}^-_r\Lambda^k(T,f), \mathcal{P}^-_r\Lambda^k(T,f) \xrightarrow[\text{tr}]{\cong} \mathring{\mathcal{P}}^-_r\Lambda^k(f) \cong \mathcal{P}_{r+k-\dim f-1}\Lambda^{\dim f-k}(f)$$

Basis for $\mathcal{P}_r^- \Lambda^k$

To create a basis for $\mathcal{P}_r^- \Lambda^k$ (the easier case), we consider all products

$$\lambda_0^{\alpha_0} \cdots \lambda_n^{\alpha_n} \phi_f, \quad f \in \Delta_k(T), \quad \sum \alpha_i = r - 1$$

This form is associated the the face whose vertices are in f or for which $\alpha_i > 0$. E.g., $\lambda_3^2 \phi_{[1,2]}$ is associated with the face $[1, 2, 3]$.

These span $\mathcal{P}_r^- \Lambda^k$. However they are not linearly independent since

$$\sum_{i=0}^{k} (-1)^i \lambda_{\sigma_i} \phi_{[\sigma_0 \cdots \widehat{\sigma}_i \cdots \sigma_k]} = 0.$$

To get a linearly independent spanning set, we impose the extra condition that if $\alpha_i \neq 0$ then $i \geq \sigma_0$ (the least vertex index of f). E.g., $\lambda_1 \lambda_2 \phi_{[1,2]}$ and $\lambda_3^2 \phi_{[1,2]}$ are included in the basis for $\mathcal{P}_3^- \Lambda^2$ but $\lambda_0 \lambda_3 \phi_{[1,2]}$ is not.

Example: explicit bases for $\mathcal{P}_r^-\Lambda^1$ and $\mathcal{P}_r^-\Lambda^2$ on a tet

$\mathcal{P}_r^-\Lambda^1(T_3)$

r	Edge $[x_i, x_j]$	Face $[x_i, x_j, x_k]$	Tet $[x_i, x_j, x_k, x_l]$
1	ϕ_{ij}		
2	$\lambda_i\phi_{ij}, \lambda_j\phi_{ij}$	$\lambda_k\phi_{ij}, \lambda_j\phi_{ik}$	
3	$\{\lambda_i^2, \lambda_j^2, \lambda_i\lambda_j\}\phi_{ij}$	$\{\lambda_i, \lambda_j, \lambda_k\}\lambda_k\phi_{ij}, \{\lambda_i, \lambda_j, \lambda_k\}\lambda_j\phi_{ik}$	$\lambda_k\lambda_l\phi_{ij}, \lambda_j\lambda_l\phi_{ik}, \lambda_j\lambda_k\phi_{il}$

$\mathcal{P}_r^-\Lambda^2(T_3)$

r	Edge $[x_i, x_j]$	Face $[x_i, x_j, x_k]$	Tet $[x_i, x_j, x_k, x_l]$
1		ϕ_{ijk}	
2		$\lambda_i\phi_{ijk}, \lambda_j\phi_{ijk}, \lambda_k\phi_{ijk}$	$\lambda_l\phi_{ijk}, \lambda_k\phi_{ijl}, \lambda_j\phi_{ikl}$
3		$\{\lambda_i^2, \lambda_j^2, \lambda_k^2\}\phi_{ijk}$	$\{\lambda_i, \lambda_j, \lambda_k, \lambda_l\}\lambda_l\phi_{ijk}$
		$\{\lambda_i\lambda_j, \lambda_i\lambda_k, \lambda_j\lambda_k\}\phi_{ijk}$	$\{\lambda_i, \lambda_j, \lambda_k, \lambda_l\}\lambda_k\phi_{ijl}$
			$\{\lambda_i, \lambda_j, \lambda_k, \lambda_l\}\lambda_j\phi_{ikl}$

Finite element differential forms on cubical meshes

The tensor product construction

Again there are two families (only?). One results from a tensor product construction. (DNA–Boffi–Bonizzoni)

Suppose we have a finite element de Rham subcomplex V on an element $S \subset \mathbb{R}^m$:

$$\cdots \to V^k \xrightarrow{d} V^{k+1} \to \cdots \qquad V^k \subset \Lambda^k(S)$$

and another, W, on another element $T \subset \mathbb{R}^n$:

$$\cdots \to W^k \xrightarrow{d} W^{k+1} \to \cdots$$

The tensor-product construction produces a new complex $V \wedge W$, a subcomplex of the de Rham complex on $S \times T$.

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$$\cdots \to W^k \xrightarrow{d} W^{k+1} \to \cdots$$

The tensor-product construction produces a new complex $V \wedge W$, a subcomplex of the de Rham complex on $S \times T$.

Shape fns: $$(V \wedge W)^k = \bigoplus_{i+j=k} \pi_S^* V^i \wedge \pi_T^* W^j \qquad (\pi_S : S \times T \to S)$$

DOFs: $$(\eta \wedge \rho)(\pi_S^* v \wedge \pi_T^* w) := \eta(v)\rho(w)$$

Finite element differential forms on cubes: the $\mathcal{Q}_r^-\Lambda^k$ family

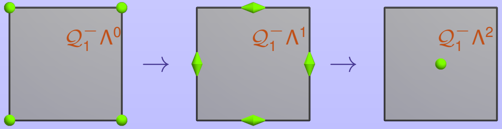
Start with the simple 1-D degree r finite element de Rham complex, V_r:

$$0 \to \mathcal{P}_r\Lambda^0(I) \xrightarrow{d} \mathcal{P}_{r-1}\Lambda^1(I) \to 0$$

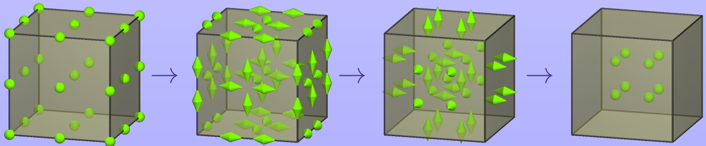
$$u(x) \quad \to \quad u'(x)\,dx$$

Take tensor product n times: $\mathcal{Q}_r^-\Lambda^k(I^n) := (V_r \wedge \cdots \wedge V_r)^k$

$$\mathcal{Q}_r = \mathcal{P}_r \otimes \mathcal{P}_r, \quad \mathcal{P}_{r-1} \otimes \mathcal{P}_r\, dx_1 + \mathcal{P}_r \otimes \mathcal{P}_{r-1}\, dx_2, \quad \mathcal{P}_{r-1} \otimes \mathcal{P}_{r-1}\, dx_1 \wedge dx_2$$



Raviart-Thomas '76



Nedelec '80

The 2nd family of finite element differential forms on cubes

The $\mathcal{S}_r\Lambda^k(I^n)$ family of FEDFs: (DNA–Awanou '12)

Shape fns:

For a form monomial $m = x_1^{\alpha_i} \cdots x_n^{\alpha_n}\, dx_{\sigma_1} \wedge \cdots \wedge dx_{\sigma_k}$, define
$\deg m = \sum \alpha_i$, $\operatorname{ldeg} m = \#\{\, i \mid \alpha_i = 1,\ \alpha_i \neq \{\sigma_1, \ldots, \sigma_k\}\,\}$.
Ex: If $m = x_1 x_2 x_3^5 dx_1$, $\deg m = 7$, $\operatorname{ldeg} m = 1$.

$$\mathcal{H}_{r,\ell}\Lambda^k(I^n) = \text{ span of monomials with deg} = r,\ \text{ldeg} \geq \ell,$$

$$\mathcal{J}_r\Lambda^k(I^n) = \bigoplus_{\ell \geq 1} \kappa\mathcal{H}_{r+\ell-1,\ell}\Lambda^{k+1}(I^n),$$

$$\mathcal{S}_r\Lambda^k(I^n) = \mathcal{P}_r\Lambda^k(I^n) \oplus \mathcal{J}_r\Lambda^k(I^n) \oplus d\mathcal{J}_{r+1}\Lambda^{k-1}(I^n).$$

DOFs: $u \mapsto \int_f u \wedge q, \quad q \in \mathcal{P}_{r-2d}\Lambda^{d-k}(f),\ f \in \Delta(I^n)$

Key properties

For any $n \geq 1$, $r \geq 1$, $0 \leq k \leq n$:

Degree property: $\mathcal{P}_r\Lambda^k(I^n) \subset \mathcal{S}_r\Lambda^k(I^n) \subset \mathcal{P}_{r+n-k}\Lambda^k(I^n)$

Inclusion property: $\mathcal{S}_r\Lambda^k(I^n) \subset \mathcal{S}_{r+1}\Lambda^k(I^n)$

Trace property: For each face f of I^n, $\quad \operatorname{tr}_f \mathcal{S}_r\Lambda^k(I^n) = \mathcal{S}_r\Lambda^k(f)$.

Subcomplex property: $d\mathcal{S}_r\Lambda^k(I^n) \subset \mathcal{S}_{r-1}\Lambda^{k+1}(I^n)$

Unisolvence: The indicated DOFs are correct in number and are unisolvent.

Commuting projections: The DOFs determine commuting projections from the de Rham complex to the subcomplex

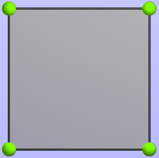
$$\mathcal{S}_r\Lambda^0(I^n) \xrightarrow{d} \mathcal{S}_{r-1}\Lambda^1(I^n) \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{S}_{r-n}\Lambda^n(I^n).$$

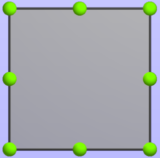
The case of 0-forms (H^1 elements)

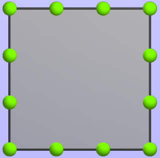
Define sdeg m of a monomial m to be the degree ignoring variables that enter linearly: sdeg $x^3yz^2 = 5$. For a polynomial p, sdeg p is the maximum over its monomials.

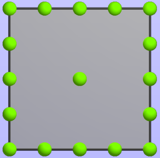
$$\mathcal{S}_r(I^n) = \{\, p \in \mathcal{P}(I^n) \mid \operatorname{sdeg} p \le r \,\}$$ DNA-Awanou ’10

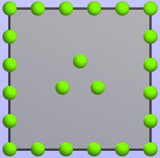
1D: $\mathcal{S}_r(I) = \mathcal{P}_r(I)$, 2D: $\mathcal{S}_r(I^2) = \mathcal{P}_r(I^2) + \operatorname{span}[x^r y, xy^r]$ serendipity


$\mathcal{S}_1(I^2)$

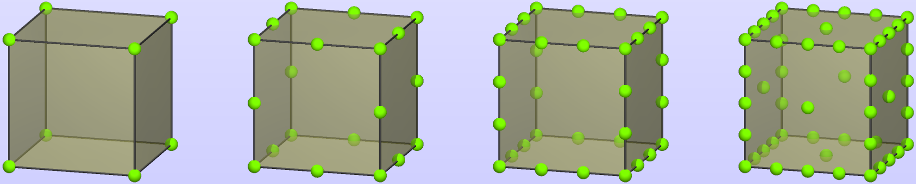

$\mathcal{S}_2(I^2)$


$\mathcal{S}_3(I^2)$


$\mathcal{S}_4(I^2)$


$\mathcal{S}_5(I^2)$

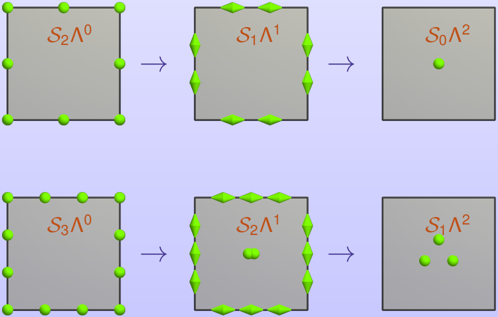
Serendipity 0-forms in more dimensions



Dimensions

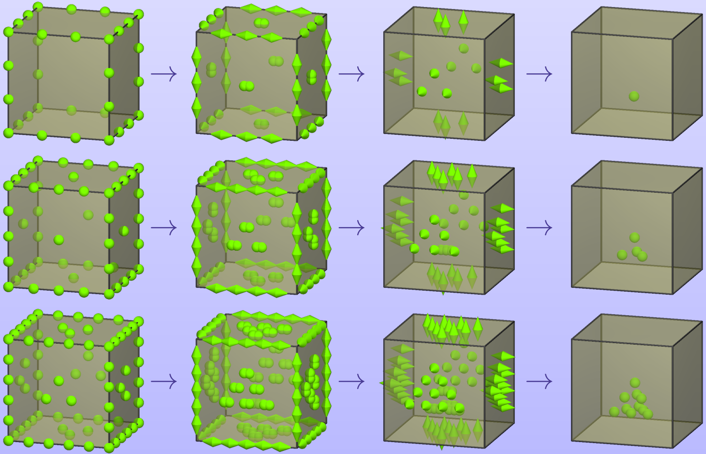
	$\mathcal{P}_r(I^n)$					$\mathcal{S}_r(I^n)$					$\mathcal{Q}_r(I^n)$				
n	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5
1	2	3	4	5	6	2	3	4	5	6	2	3	4	5	6
2	3	6	10	15	21	4	8	12	17	23	4	9	16	25	36
3	4	10	20	35	56	8	20	32	50	74	8	27	64	125	216
4	5	15	35	70	126	16	48	80	136	216	16	81	256	625	1296

The 2nd cubic family in 2-D



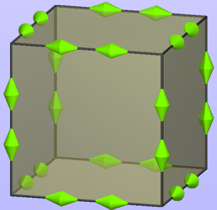
	$\mathcal{S}_r\Lambda^k(I^2)$				
k	1	2	3	4	5
0	4	8	12	17	23
1	8	14	22	32	44
2	3	6	10	15	21

The 2nd cubic family in 3-D



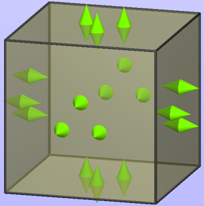
Dimensions and low order cases

	$\mathcal{S}_r\Lambda^k(I^3)$				
k	1	2	3	4	5
0	8	20	32	50	74
1	24	48	84	135	204
2	18	39	72	120	186
3	4	10	20	35	56



$\mathcal{S}_1\Lambda^1(I^3)$

new element



$\mathcal{S}_1\Lambda^2(I^3)$

corrected element

The 3D shape functions in traditional FE language

$\mathcal{S}_r\Lambda^0$: polynomials u such that $\operatorname{sdeg} u \le r$

$\mathcal{S}_r\Lambda^1$:

$$(v_1, v_2, v_3) + (x_2x_3(w_2 - w_3), x_3x_1(w_3 - w_1), x_1x_2(w_1 - w_2)) + \operatorname{grad} u,$$

$v_i \in \mathcal{P}_r$, $w_i \in \mathcal{P}_{r-1}$ independent of x_i, $\operatorname{sdeg} u \le r + 1$

$\mathcal{S}_r\Lambda^2$:

$$(v_1, v_2, v_3) + \operatorname{curl}(x_2x_3(w_2 - w_3), x_3x_1(w_3 - w_1), x_1x_2(w_1 - w_2)),$$

$v_i, w_i \in \mathcal{P}_r(I^3)$ with w_i independent of x_i

$\mathcal{S}_r\Lambda^3$: $v \in \mathcal{P}_r$