Finite Element Exterior Calculus and Applications

Part IV

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Finite element differential forms on cubical meshes

References

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The tensor product construction

DNA–Boffi–Bonizzoni 2012

Suppose we have a de Rham subcomplex V on an element $S \subset \mathbb{R}^m$:

$$\cdots \to V^k \xrightarrow{d} V^{k+1} \to \cdots \qquad V^k \subset H\Lambda^k(S)$$

and another, W, on another element $T \subset \mathbb{R}^n$:

$$\cdots \to W^k \xrightarrow{d} W^{k+1} \to \cdots$$

The tensor-product construction produces a new complex $V \wedge W$, a subcomplex of the de Rham complex on $S \times T$.

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Shape fns: $$(V \wedge W)^k = \bigoplus_{i+j=k} \pi_S^* V^i \wedge \pi_T^* W^j \qquad (\pi_S : S \times T \to S)$$

DOFs: $$(\eta \wedge \rho)(\pi_S^* v \wedge \pi_T^* w) := \eta(v)\rho(w)$$

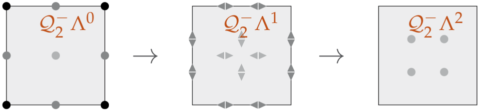
Start with the simple 1-D degree r finite element de Rham complex, V_r:

$$0 \to \mathcal{P}_r\Lambda^0(I) \xrightarrow{d} \mathcal{P}_{r-1}\Lambda^1(I) \to 0$$

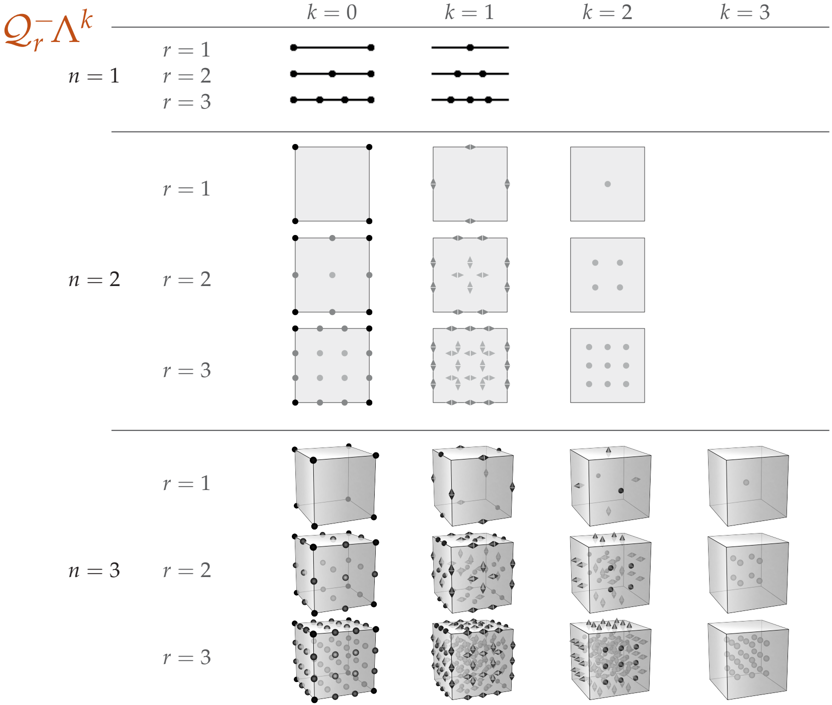
$$u(x) \to u'(x)\,dx$$

Take tensor product n times: $\mathcal{Q}_r^-\Lambda^k(I^n) := (V_r \wedge \cdots \wedge V_r)^k$

$$\mathcal{Q}_r^-\Lambda^0 = \mathcal{Q}_r,$$
$$\mathcal{Q}_r^-\Lambda^1 = \mathcal{Q}_{r-1,r,r,\ldots}dx^1 + \mathcal{Q}_{r,r-1,r,\ldots}dx^2 + \cdots,$$
$$\mathcal{Q}_r^-\Lambda^2 = \mathcal{Q}_{r-1,r-1,r,\ldots}dx^1 \wedge dx^2 + \cdots, \quad \ldots$$



constant degree



$\mathcal{Q}_r^- \Lambda^k$
$k = 0$
$k = 1$
$k = 2$
$k = 3$
$n = 1$
$n = 2$
$n = 3$
$r = 1$
$r = 2$
$r = 3$

The 2nd family on cubes: 0-forms

DNA–Awanou 2011

The $\mathcal{Q}_r^- \Lambda^k$ family reduces to $\mathcal{Q}_r$ when $k = 0$. For the second family, we get the serendipy space $\mathcal{S}_r$.

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2-D shape fns: $\mathcal{S}_r(I^2) = \mathcal{P}_r(I^2) \oplus \text{span}[x_1^r x_2, x_1 x_2^r]$

DOFs: $u \mapsto \int_f \text{tr}_f\, u\, q, \quad q \in \mathcal{P}_{r-2d}(f), f \in \Delta_d(I^n)$

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DOFs: $u \mapsto \int_f \operatorname{tr}_f u\, q, \quad q \in \mathcal{P}_{r-2d}(f), f \in \Delta_d(I^n)$

n-D shape fns: $\mathcal{S}_r(I^n) = \mathcal{P}_r(I^n) \oplus \bigoplus_{\ell \ge 1} \mathcal{H}_{r+\ell,\ell}(I^n)$

$\mathcal{H}_{r,\ell}(I^n) =$ span of monomials of degree r, linear in $\ge \ell$ variables

The 2nd family of finite element differential forms on cubes

DNA–Awanou 2012

The $\mathcal{S}_r\Lambda^k(I^n)$ family of FEDFs, uses the serendipity spaces for 0-forms, and serendipity-like DOFs.

DOFs: $u \mapsto \int_f \mathrm{tr}_f\, u \wedge q, \quad q \in \mathcal{P}_{r-2(d-k)}\Lambda^{d-k}(f), f \in \Delta_d(I^n), d \geq k$

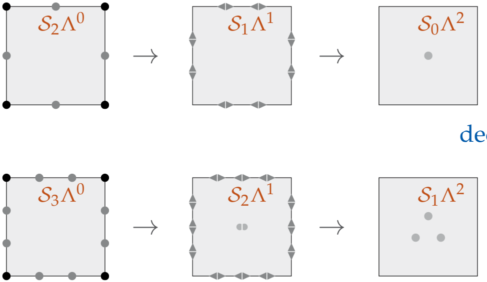
Shape fns:

$$\mathcal{S}_r\Lambda^k(I^n) = \mathcal{P}_r\Lambda^k(I^n) \oplus \bigoplus_{\ell \geq 1} \underbrace{[\kappa\mathcal{H}_{r+\ell-1,\ell}\Lambda^{k+1}(I^n) \oplus d\kappa\mathcal{H}_{r+\ell,\ell}\Lambda^k(I^n)]}_{\deg = r+\ell}$$

$$\mathcal{H}_{r,\ell}\Lambda^k(I^n) = \text{ span of monomials } x_1^{\alpha_i} \cdots x_n^{\alpha_n}\, dx_{\sigma_1} \wedge \cdots \wedge dx_{\sigma_k}, \\ |\alpha| = r, \text{linear in} \geq \ell \text{ variables not counting the } x_{\sigma_i}$$

Unisolvence holds for all $n \geq 1, r \geq 1, 0 \leq k \leq n$.

The 2nd cubic family in 2-D



decreasing degree

$\mathcal{S}_r\Lambda^k(I^2)$

k	1	2	3	4	5
0	4	8	12	17	23
1	8	14	22	32	44
2	3	6	10	15	21

$\mathcal{Q}_r^-\Lambda^k(I^2)$

k	1	2	3	4	5
0	4	9	16	25	36
1	4	12	24	40	60
2	1	4	9	16	25

The 3D shape functions in traditional FE language

$\mathcal{S}_r\Lambda^0$: polynomials u such that $\deg u \le r + \operatorname{ldeg} u$

$\mathcal{S}_r\Lambda^1$:

$$(v_1, v_2, v_3) + (x_2x_3(w_2 - w_3), x_3x_1(w_3 - w_1), x_1x_2(w_1 - w_2)) + \operatorname{grad} u,$$

$v_i \in \mathcal{P}_r$, $w_i \in \mathcal{P}_{r-1}$ independent of x_i, $\deg u \le r + \operatorname{ldeg} u + 1$

$\mathcal{S}_r\Lambda^2$:

$$(v_1, v_2, v_3) + \operatorname{curl}(x_2x_3(w_2 - w_3), x_3x_1(w_3 - w_1), x_1x_2(w_1 - w_2)),$$

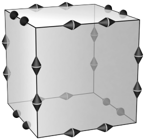
$v_i, w_i \in \mathcal{P}_r(I^3)$ with w_i independent of x_i

$\mathcal{S}_r\Lambda^3$: $v \in \mathcal{P}_r$

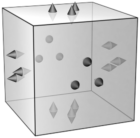
Dimensions and low order cases

k	$\mathcal{S}_r\Lambda^k(I^3)$ 1	2	3	4	5
0	8	20	32	50	74
1	24	48	84	135	204
2	18	39	72	120	186
3	4	10	20	35	56

k	$\mathcal{Q}_r^-\Lambda^k(I^3)$ 1	2	3	4	5
0	8	27	64	125	216
1	12	54	96	200	540
2	6	36	108	240	450
3	1	8	27	64	125



$\mathcal{S}_1\Lambda^1(I^3)$
new element

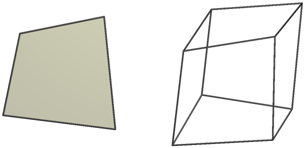


$\mathcal{S}_1\Lambda^2(I^3)$
corrected element

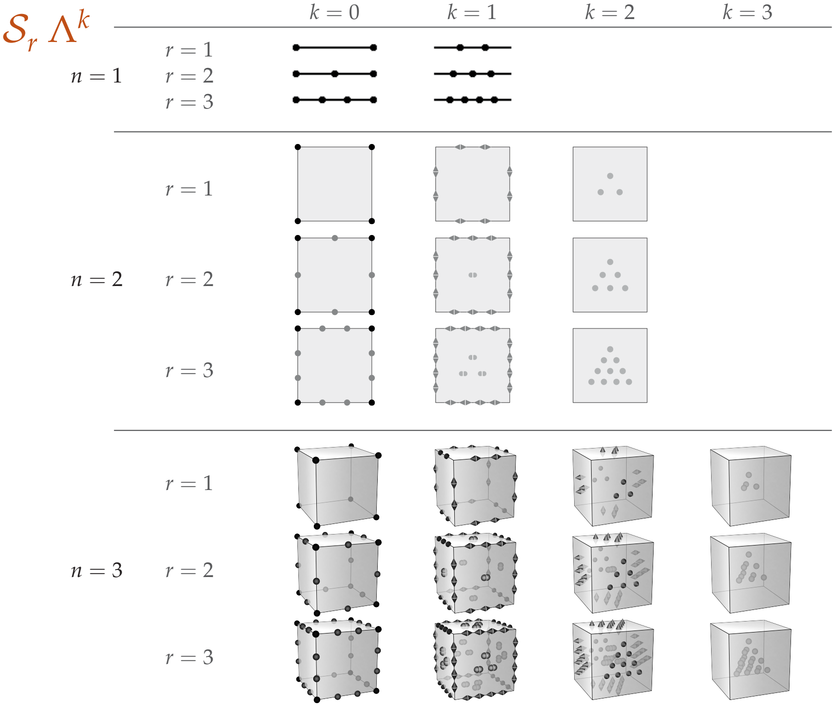
Approximation properties

On cubes the $\mathcal{Q}_r^- \Lambda^k$ and $\mathcal{S}_r^- \Lambda^k$ spaces provide the expected order of approximation. Same is true on parallelotopes, but accuracy is lost by non-affine distortions, *with greater loss, the greater the form degree k*.

- The L^2 approximation rate of the space $\mathcal{Q}_r = \mathcal{Q}_r^- \Lambda^0$ is $r+1$ on either affinely or multilinearly mapped elements.
- The rate for $\mathcal{S}_r = \mathcal{S}_r \Lambda^0$ is $r+1$ on affinely mapped elements, but only $\max(2, \lfloor r/n \rfloor + 1)$ on multilinearly mapped elements.
- The rate for $\mathcal{Q}_r^- \Lambda^k$, $k > 0$, is r on affinely mapped elements, $r - k + 1$ on multilinearly mapped elements.
- The rate for $\mathcal{P}_r \Lambda^n = \mathcal{S}_r \Lambda^n$ is $r+1$ for affinely mapped elements, $\lfloor r/n \rfloor - n + 2$ for multilinearly mapped.



DNA-Boffi-Bonizzoni 2012



$\mathcal{S}_r \Lambda^k$
$k = 0$
$k = 1$
$k = 2$
$k = 3$
$n = 1$
$n = 2$
$n = 3$
$r = 1$
$r = 2$
$r = 3$

Periodic Table of the Finite Elements

k = 0 | k = 1 | k = 2 | k = 3

n = 1

r = 1, r = 2, r = 3

$\mathcal{P}_r^-\Lambda^k$

The shape function space for $\mathcal{P}_r^-\Lambda^k$ is $\mathcal{P}_{r-1}\Lambda^k + \kappa\mathcal{P}_{r-1}\Lambda^{k+1}$, where κ is the Koszul differential.[7] It includes the full polynomial space $\mathcal{P}_{r-1}\Lambda^k$, is included in $\mathcal{P}_r\Lambda^k$, and has dimension

$$\dim \mathcal{P}_r^-\Lambda^k(\Delta_n) = \binom{r+n}{r+k}\binom{r+k-1}{k}.$$

The degrees of freedom are given on faces f of dimension $d \ge k$ by moments of the trace weighted by a full polynomial space:

$$u \mapsto \int_f (\operatorname{tr}_f u)\wedge q, \quad q \in \mathcal{P}_{r+k-d-1}\Lambda^{d-k}(f).$$

The spaces with constant degree r form a complex:

$$\mathcal{P}_r^-\Lambda^0 \xrightarrow{d} \mathcal{P}_r^-\Lambda^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{P}_r^-\Lambda^n.$$

$\mathcal{P}_r\Lambda^k$

The shape function space for $\mathcal{P}_r\Lambda^k$ consists of all differential k-forms with polynomial coefficients of degree at most r, and has dimension

$$\dim \mathcal{P}_r\Lambda^k(\Delta_n) = \binom{r+n}{r+k}\binom{r+k}{k}.$$

The degrees of freedom are given on faces f of dimension $d \ge k$ by moments of the trace weighted by a $\mathcal{P}_r^-$ space:

$$u \mapsto \int_f (\operatorname{tr}_f u)\wedge q, \quad q \in \mathcal{P}^-_{r+k-d}\Lambda^{d-k}(f).$$

The spaces with decreasing degree r form a complex:

$$\mathcal{P}_r\Lambda^0 \xrightarrow{d} \mathcal{P}_{r-1}\Lambda^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{P}_{r-n}\Lambda^n.$$

$\mathcal{Q}_r^-\Lambda^k$

This family is constructed from the complex of 1-dimensional finite elements using a tensor product construction.[10] The shape function space on the unit cube $\square_n = I^n$ is given by

$$\bigoplus_{\sigma\in\Sigma(k,n)} \left[\bigotimes_{i=1}^{n} \mathcal{P}_{r-\delta_{i,\sigma}}(I)\right] dx^{\sigma_1}\wedge\cdots\wedge dx^{\sigma_k},$$

where $\Sigma(k,n)$ denotes the increasing maps $\{1,\ldots,k\}\to\{1,\ldots,n\}$. Its dimension is $\dim \mathcal{Q}_r^-\Lambda^k(\square_n) = \binom{n}{k} r^k (r+1)^{n-k}$. The degrees of freedom are

$$u \mapsto \int_f (\operatorname{tr}_f u)\wedge q, \quad q \in \mathcal{Q}^-_{r-1}\Lambda^d(f).$$

The spaces with constant degree r form a complex:

$$\mathcal{Q}_r^-\Lambda^0 \xrightarrow{d} \mathcal{Q}_r^-\Lambda^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{Q}_r^-\Lambda^n.$$

$\mathcal{S}_r\Lambda^k$

The shape function space for $\mathcal{S}_r\Lambda^k$ is given by

$$\mathcal{P}_r\Lambda^k \oplus \bigoplus_{\ell\ge 1} \left[\kappa\mathcal{H}_{r+\ell-1,\ell}\Lambda^{k+1} \oplus d\kappa\mathcal{H}_{r+\ell,\ell}\Lambda^k\right],$$

where $\mathcal{H}_{r,\ell}\Lambda^k$ consists of homogeneous polynomial k-forms of degree r which are linear and undifferentiated in at least ℓ variables.[11] Its dimension is $\dim \mathcal{S}_r\Lambda^k(\square_n) = \sum_{d\ge k} 2^{n-d}\binom{n}{d}\binom{r-d+2k}{d}\binom{d}{k}$. The degrees of freedom are

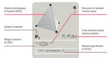
$$u \mapsto \int_f (\operatorname{tr}_f u)\wedge q, \quad q \in \mathcal{P}_{r-2(d-k)}\Lambda^{d-k}(f).$$

The spaces with decreasing degree r form a complex:

$$\mathcal{S}_r\Lambda^0 \xrightarrow{d} \mathcal{S}_{r-1}\Lambda^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{S}_{r-n}\Lambda^n.$$

n = 2, **n = 3**

Legend



$$\dim \mathcal{P}_r^-\Lambda^k(\Delta_n) = \binom{r+n}{r+k}\binom{r+k-1}{k}$$

	k	r = 1	2	3	4	5	6	7
n = 1	0	2	3	4	5	6	7	8
	1	1	2	3	4	5	6	7
n = 2	0	3	6	10	15	21	28	36
	1	3	8	15	24	35	48	63
	2	1	3	6	10	15	21	28
n = 3	0	4	10	20	35	56	84	120
	1	6	20	45	84	140	216	315
	2	4	15	36	70	120	189	280
	3	1	4	10	20	35	56	84
n = 4	0	5	15	35	70	126	210	330
	1	10	40	105	224	420	720	1155
	2	10	45	126	280	540	945	1540
	3	5	24	70	160	315	560	924
	4	1	5	15	35	70	126	210

$$\dim \mathcal{P}_r\Lambda^k(\Delta_n) = \binom{r+n}{r+k}\binom{r+k}{k}$$

	k	r = 1	2	3	4	5	6	7
n = 1	0	2	3	4	5	6	7	8
	1	2	3	4	5	6	7	8
n = 2	0	3	6	10	15	21	28	36
	1	6	12	20	30	42	56	72
	2	3	6	10	15	21	28	36
n = 3	0	4	10	20	35	56	84	120
	1	12	30	60	105	168	252	360
	2	12	30	60	105	168	252	360
	3	4	10	20	35	56	84	120
n = 4	0	5	15	35	70	126	210	330
	1	20	60	140	280	504	840	1320
	2	30	90	210	420	756	1260	1980
	3	20	60	140	280	504	840	1320
	4	5	15	35	70	126	210	330

$$\dim \mathcal{Q}_r^-\Lambda^k(\square_n) = \binom{n}{k} r^k (r+1)^{n-k}$$

	k	r = 1	2	3	4	5	6	7
n = 1	0	2	3	4	5	6	7	8
	1	1	2	3	4	5	6	7
n = 2	0	4	9	16	25	36	49	64
	1	4	12	24	40	60	84	112
	2	1	4	9	16	25	36	49
n = 3	0	8	27	64	125	216	343	512
	1	12	54	144	300	540	882	1344
	2	6	36	108	240	450	756	1176
	3	1	8	27	64	125	216	343
n = 4	0	16	81	256	625	1296	2401	4096
	1	32	216	768	2000	4320	8232	14336
	2	24	216	864	2400	5400	10584	18816
	3	8	96	432	1280	3000	6048	10976
	4	1	16	81	256	625	1296	2401

$$\dim \mathcal{S}_r\Lambda^k(\square_n) = \sum_{d=k}^{\min(n,\lfloor r/2\rfloor+k)} 2^{n-d}\binom{n}{d}\binom{r-d+2k}{d}\binom{d}{k}$$

	k	r = 1	2	3	4	5	6	7
n = 1	0	2	3	4	5	6	7	8
	1	2	3	4	5	6	7	8
n = 2	0	4	8	12	17	23	30	38
	1	8	14	22	32	44	58	74
	2	3	6	10	15	21	28	36
n = 3	0	8	20	32	50	74	105	144
	1	24	48	84	135	204	294	408
	2	18	36	72	120	186	273	384
	3	4	10	20	35	56	84	120
n = 4	0	16	48	80	136	216	328	480
	1	64	144	272	472	758	1188	1764
	2	72	168	336	606	1014	1602	2418
	3	32	84	180	340	588	952	1464
	4	5	15	35	70	126	210	330

Finite elements

The table presents the primary spaces of finite elements for the discretization of the fundamental operators of vector calculus: the gradient, curl, and divergence. A finite element space is a space of piecewise polynomial functions on a domain determined by: (1) a mesh of the domain into polyhedral cells called *elements*, (2) a finite dimensional space of polynomial functions on each element called the *shape functions*, and (3) a unisolvent set of functionals on the shape functions of each element called *degrees of freedom* (DOFs), each DOF being associated to a (generalized) face of the element, and specifying a quantity which takes a single value for all elements sharing the face. The element diagrams depict the DOFs and their association to faces.

The spaces $\mathcal{P}_r^-\Lambda^k$ and $\mathcal{P}_r\Lambda^k$ depicted on the left half of the table are the two primary families of finite element spaces for meshes of simplices, and the spaces $\mathcal{Q}_r^-\Lambda^k$ and $\mathcal{S}_r\Lambda^k$ on the right side are for meshes of cubes or boxes. Each is defined in any dimension $n \ge 1$ for each value of the polynomial degree $r \ge 1$, and each value of $0 \le k \le n$. The parameter k refers to the operator: the spaces consist of differential k-forms which belong to the domain of the kth exterior derivative. Thus for $k = 0$, the spaces discretize the Sobolev space H^1, the domain of the gradient operator; for $k = 1$, they discretize $H(\text{curl})$, the domain of the curl; for $k = n-1$ they discretize $H(\text{div})$, the domain of the divergence; and for $k = n$, they discretize L^2.

The spaces $\mathcal{P}_r^-\Lambda^0$ and $\mathcal{P}_r\Lambda^0$, which coincide, are the earliest finite elements, going back in the case $r = 1$ of linear elements to *Courant*,[1] and collectively referred to as *the Lagrange elements*. The spaces $\mathcal{P}_{r+1}^-\Lambda^n$ and $\mathcal{P}_r\Lambda^n$, which also coincide, are the discontinuous *Galerkin elements*, consisting of piecewise polynomials with no interelement continuity imposed, first introduced by *Reed and Hill*.[2] The space $\mathcal{P}_r^-\Lambda^1$ in 2 dimensions was introduced by *Raviart and Thomas*[3] and generalized to the 3-dimensional spaces $\mathcal{P}_r^-\Lambda^1$ and $\mathcal{P}_r^-\Lambda^2$ by *Nédélec*,[4] while $\mathcal{P}_r\Lambda^1$ is due to *Brezzi, Douglas and Marini*[5] in 2 dimensions, its generalization to 3 dimensions again due to *Nédélec*.[6] The unified treatment and notation of the $\mathcal{P}_r^-\Lambda^k$ and $\mathcal{P}_r\Lambda^k$ families is due to *Arnold, Falk and Winther* as part of *finite element exterior calculus*,[7] extending earlier work of *Hiptmair* for the $\mathcal{P}_r^-\Lambda^k$ family.[8] The space $\mathcal{P}_1^-\Lambda^k$ is the span of the *elementary forms* introduced by *Whitney*.[9]

The family $\mathcal{Q}_r^-\Lambda^k$ of cubical elements can be derived from the 1-dimensional Lagrange and discontinuous Galerkin elements by a tensor product construction detailed by *Arnold, Boffi and Bonizzoni*,[10] but for the most part were presented individually along with the corresponding simplicial elements in the papers mentioned. The second cubical family $\mathcal{S}_r\Lambda^k$ is due to *Arnold and Awanou*.[11]

The finite elements in this table have been implemented as part of the *FEniCS Project*.[12,13,14] Each may be referenced in the *Unified Form Language* (UFL)[15] by giving its family, shape, and degree, with the family as shown on the table. For example, the space $\mathcal{P}_3\Lambda^1(\Delta_3)$ may be referred to in UFL as:

```
FiniteElement("N2E", tetrahedron, 3)
```

Alternatively, the elements may be accessed in a uniform fashion as:

```
FiniteElement("P-", shape, r, k)
FiniteElement("P", shape, r, k)
FiniteElement("Q-", shape, r, k)
FiniteElement("S", shape, r, k)
```

for $\mathcal{P}_r^-\Lambda^k$, $\mathcal{P}_r\Lambda^k$, $\mathcal{Q}_r^-\Lambda^k$, and $\mathcal{S}_r\Lambda^k$, respectively.

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[simula . research laboratory] - by thinking constantly about it

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Periodic Table of the Finite Elements